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An Ultra-Weak Discontinuous Galerkin Method with Implicit–Explicit Time-Marching for Generalized Stochastic KdV Equations

Yunzhang Li[1] · **Chi-Wang Shu**[2] · **Shanjian Tang**[1]

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Abstract

In this paper, an ultra-weak discontinuous Galerkin (DG) method is developed to solve the generalized stochastic Korteweg–de Vries (KdV) equations driven by a multiplicative temporal noise. This method is an extension of the DG method for purely hyperbolic equations and shares the advantage and flexibility of the DG method. Stability is analyzed for the general nonlinear equations. The ultra-weak DG method is shown to admit the optimal error of order $k+1$ in the sense of the spatial $L^2(0, 2\pi)$-norm for semi-linear stochastic equations, when polynomials of degree $k \geq 2$ are used in the spatial discretization. A second order implicit–explicit derivative-free time discretization scheme is also proposed for the matrix-valued stochastic ordinary differential equations derived from the spatial discretization. Numerical examples using Monte Carlo simulation are provided to illustrate the theoretical results.

Keywords Ultra-weak discontinuous Galerkin method · Generalized stochastic KdV equations · Multiplicative stochastic noise · Stability analysis · Error estimates · Implicit–explicit time discretization

Mathematics Subject Classification 65C30 · 60H35

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✉ Yunzhang Li
15110180027@fudan.edu.cn

Chi-Wang Shu
chi-wang_shu@brown.edu

Shanjian Tang
sjtang@fudan.edu.cn

[1] Department of Finance and Control Sciences, School of Mathematical Sciences, Fudan University, Shanghai 200433, China

[2] Division of Applied Mathematics, Brown University, Providene, RI 02912, USA

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1 Introduction

The Korteweg–de Vries (KdV) equations were introduced in 1895 by Korteweg and de Vries [20] to model long, unidirectional, dispersive waves of small amplitude. It was generalized to study the nonlinear anharmonic lattices [34]. The equations turn out to be not only good models for water waves, but also very useful approximation models in nonlinear studies which incorporate and balance a weak nonlinearity and weak dispersive effects. The stochastic KdV equations arise in the propagation of weakly nonlinear waves in a noisy plasma [4,18,30]. It is also of interest in any circumstances when the KdV equations are used, since the stochastic forcing may represent terms that have been neglected in the derivation of this ideal model. In this paper we present an ultra-weak discontinuous Galerkin (DG) method for the following stochastic generalized KdV equation with a periodic boundary condition and a multiplicative temporal noise:

$$\begin{cases} du = -[u_{xxx} + f(u)_x]\, dt + g(\cdot, x, t, u)\, dW_t, & (x, t) \in [0, 2\pi] \times (0, T]; \\ u(x, 0) = u_0(x), & x \in [0, 2\pi], \end{cases} \tag{1.1}$$

where the terminal time $T > 0$ is a fixed real number, and $\{W_t, 0 \le t \le T\}$ is a standard one-dimensional Brownian motion on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\{\mathcal{F}_t, 0 \le t \le T\}$ the augmented natural filtration of W. We make the following hypotheses:

(H1) The initial condition $u_0 \in L^2(0, 2\pi)$.

(H2) The functions f and g are locally Lipschitz continuous, i.e., for any $M \in \mathbb{N}_+$, there exists a positive constant $L(M)$ such that, for all $(\omega, x, t) \in \Omega \times [0, 2\pi] \times [0, T]$ and all $(u, u') \in \mathbb{R}^2$ with $|u| \vee |u'| \le M$,

$$\left| f(u) - f(u') \right| \vee \left| g(\omega, x, t, u) - g(\omega, x, t, u') \right| \le L(M) \left| u - u' \right|.$$

(H3) The functions f and g are at most linearly growing, i.e. there exists a constant $C > 0$ such that for any $(\omega, x, t, u) \in \Omega \times [0, 2\pi] \times [0, T] \times \mathbb{R}$,

$$|f(u)| \vee |g(\omega, x, t, u)| \le C(1 + |u|).$$

The existence and uniqueness of solutions for the stochastic KdV equations with a multiplicative stochastic forcing term involving a temporal white noise was established by de Bouard and Debussche [15] (cf. also [13,16–18] and the references therein). In most cases, it is not possible to have explicit solutions to these problems. Thus numerical solutions of these stochastic partial differential equations (SPDEs) naturally receive a lot of attention.

Concerning numerical schemes for stochastic KdV equations, Debussche and Printems [14] numerically investigated the influence of an additive noise on the evolution of solutions based on finite elements and least-squares. By a modified Zabusky–Kruskal finite difference scheme, Rose [29] discussed the large time behavior of the stochastic KdV equations and verified the diffusion of solitons. Lin et al. [23] gave numerical solutions of the stochastic KdV equations for the three cases of additive time-dependent noise, multiplicative space-dependent noise, and a combination of both, but lacked of any result on stability and error. They employed polynomial chaos for discretization in random space, and local discontinuous Galerkin (LDG) and finite difference for discretization in the physical space. Unlike the plethora of the theoretical and perturbation-based works, little attention seemed to be paid to the stability and error of the high-order approximation schemes for stochastic KdV equations, which are the main objective of our current paper.

The first DG method was presented by Reed and Hill [28] for a deterministic time-independent linear hyperbolic equation in the framework of neutron transport. A major development of the DG method is the Runge–Kutta DG (RKDG) framework introduced for nonlinear hyperbolic conservation laws of first order spatial derivatives in a series of papers by Cockburn et al. [7–11]. Subsequently, the method was extended to partial differential equations of order higher than one (e.g. [2,5,12,33]).

In this paper, we extend the ultra-weak DG method to stochastic generalized KdV equations (1.1). The ultra-weak DG method refers to the DG method [31] in which the integration by parts formula is used repeatedly to transfer all the spatial derivatives from the solution to the test function in the weak formulations. It can be dated back at least to [3]. In [5], Cheng and Shu developed an ultra-weak DG method for general time-dependent problems with higher order spatial derivatives, which can be used to numerically solve the deterministic generalized KdV equations. They obtained the L^2-norm stability results by carefully choosing the numerical fluxes resulting from integration by parts. With the help of the local Gauss–Radau projection, they proved error estimates for nonlinear deterministic equation. Our numerical scheme is the stochastic counterpart of the above work and shares the following advantages and flexibilities of the classical DG method: (1) it is easy to design high order approximations, thus allowing efficient p-adaptivity; (2) it is flexible on complicated geometries, thus allowing efficient h-adaptivity; (3) it is local in data communications, thus allowing efficient parallel implementations.

There are also some types of DG methods for SPDEs (see [22] and the references therein). Recently, Li et al. proposed a DG method [21] for nonlinear stochastic hyperbolic conservation laws and an LDG method [22] for nonlinear parabolic SPDEs. By estimating the quadratic variation process of the approximate solution, they investigated the stability for fully nonlinear equations and the error estimates for semi-linear equations. Motivated by these earlier results, in this paper we study the stability for nonlinear KdV equations and error estimates for semi-linear third-order SPDEs.

The ultra-weak DG method is a scheme for spatial discretization, which needs to be coupled with a high-order time discretization. The explicit methods used in [21,22] are efficient for solving first-order SPDEs and are tolerable for second-order SPDEs. However, since the KdV equations contain third-order spatial derivative, explicit time discretization will suffer from a stringent time-step restriction $\Delta t \sim (\Delta x)^3$ for stability. Thus it is natural to consider an implicit time-marching to get rid of this time-step restriction. In many applications, the convection terms $f(\cdot)$ are often nonlinear; hence we would like to treat them explicitly while using implicit time discretization only for the third-order term in the KdV equations. Such time discretizations are called implicit–explicit (IMEX) time discretizations [1]. Wang et al. [32] proposed an IMEX time discretization scheme for LDG method, which is unconditionally stable for the nonlinear problems. Inspired by them, we give an implementable second order time discretization for the matrix-valued SDE (6.1), which coincides with the one for ODEs in [32] for the degenerate case that $b(\cdot) \equiv 0$.

The paper is organized as follows. In Sect. 2, we introduce notations, definitions and auxiliary results used in the paper. In Sect. 3, we present the ultra-weak DG method for nonlinear KdV equations (1.1), and study the existence and uniqueness of the solution to the stochastic differential equations (SDEs) derived from the spatial discretization. In Sect. 4, we investigate the L^2-stability for the fully nonlinear stochastic equations. In Sect. 5, we obtain the optimal error estimate ($\mathcal{O}(h^{k+1})$) for semilinear SPDEs with respect to spatial $L^2(0, 2\pi)$-norm. In Sect. 6, we establish a second-order IMEX derivative-free time discretization for matrix-valued SDEs to collaborate with the semi-discrete ultra-weak DG scheme. Finally in Sect. 7 the paper ends with a series of numerical experiments on some model problems, which confirm our analytical results.

2 Notations, Definitions and Auxiliary Results

In this section, we introduce notations, definitions, and some auxiliary results.

2.1 Notations

We denote the mesh by $I_j = \left[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right]$, for $j = 1, \ldots, N$. The mesh size is denoted by $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, with $h = \max\limits_{1\leq j\leq N} h_j$ being the maximum mesh size. We assume that the mesh is regular, namely the ratio between the maximum and the minimum mesh sizes stays bounded during mesh refinements. Denote by $P^k(I)$ the totality of all polynomials on I of the degree up to k for any interval I. We define the piece-wisely polynomial space V_h as follows:

$$V_h := \left\{v : v \text{ restricted on each } I_j \text{ lies in } P^k(I_j) \text{ for } j = 1, \ldots, N\right\}.$$

Note that functions in V_h might have discontinuities on an element interface.

We denote by $\|\cdot\|$ and $\|\cdot\|_{H^{m,p}}$, the $L^2(0, 2\pi)$ norm and the Sobolev norm with respect to the spatial variable x, respectively. For simplicity, by $\|\cdot\|_{H^m}$, it means $\|\cdot\|_{H^{m,2}}$. We denote by $\mathcal{S}^p(\Omega \times [0, T]; L^2(0, 2\pi))$, the space of all adapted continuous processes $\phi : \Omega \times [0, T] \longrightarrow L^2(0, 2\pi)$ such that $\left(\mathbb{E}\left[\sup\limits_{0\leq t\leq T} \|\phi(t)\|^p\right]\right)^{\frac{1}{p}} < \infty$. An element of $\mathbb{R}^{k\times d}$ is a $k \times d$ matrix, and its Euclidean norm is given by $|y| := \sqrt{trace(yy^*)}$ for $y \in \mathbb{R}^{k\times d}$.

The solution of the numerical scheme is denoted by u_h, which belongs to the finite element space V_h. Set $u^+_{j+\frac{1}{2}} := u(x^+_{j+\frac{1}{2}})$ and $u^-_{j+\frac{1}{2}} := u(x^-_{j+\frac{1}{2}})$, with $x^\pm_{j+\frac{1}{2}} := x_{j+\frac{1}{2}}\pm$.

By $C > 0$, we denote a generic constant, which in particular does not depend on the discretization width h and possibly changes from line to line. Since the Itô integral is not defined path-wisely, the argument ω of the integrand as a stochastic process will be omitted in the rest of this paper if there is no danger of confusion.

2.2 The Numerical Flux

For the convenience of notation we would like to introduce the following numerical flux related to the ultra-weak DG spatial discretization. The given monotone numerical flux $\widehat{f}\left(q^-, q^+\right)$ depends on the two values of the function q at the discontinuity point $x_{j+\frac{1}{2}}$, namely $q^\pm_{j+\frac{1}{2}} = q\left(x^\pm_{j+\frac{1}{2}}\right)$. The numerical flux $\widehat{f}\left(q^-, q^+\right)$ satisfies the following conditions:

(a) it is locally Lipschitz continuous and linearly growing;
(b) it is consistent with the physical flux $f(q)$, i.e., $\widehat{f}(q, q) = f(q)$;
(c) it is nondecreasing in the first argument, and nonincreasing in the second argument.

2.3 Projection Properties

Consider the standard L^2-projection of a function u with $(k+1)$-th continuous derivatives into space V_h, denoted by $\mathcal{P}$, i.e., for each j,

$$\int_{I_j} [\mathcal{P}u(x) - u(x)]\, v(x)\, dx = 0, \quad \forall v \in P^k(I_j),$$

and the local Gauss-Radau projection $\mathcal{Q}$ into space V_h, which satisfies, for $k = 2$,

$$\begin{cases} n\mathcal{Q}u\left(x^-_{j+\frac{1}{2}}\right) = u\left(x^-_{j+\frac{1}{2}}\right), \\ (\mathcal{Q}u)_x\left(x^+_{j-\frac{1}{2}}\right) = u_x\left(x^+_{j-\frac{1}{2}}\right), \\ (\mathcal{Q}u)_{xx}\left(x^+_{j-\frac{1}{2}}\right) = u_{xx}\left(x^+_{j-\frac{1}{2}}\right), \end{cases}$$

and for $k \geq 3$,

$$\begin{cases} \int_{I_j} [\mathcal{Q}u(x) - u(x)]\, r(x)\, dx = 0, \quad \forall r \in P^{k-3}(I_j), \\ \mathcal{Q}u\left(x^-_{j+\frac{1}{2}}\right) = u\left(x^-_{j+\frac{1}{2}}\right), \\ (\mathcal{Q}u)_x\left(x^+_{j-\frac{1}{2}}\right) = u_x\left(x^+_{j-\frac{1}{2}}\right), \\ (\mathcal{Q}u)_{xx}\left(x^+_{j-\frac{1}{2}}\right) = u_{xx}\left(x^+_{j-\frac{1}{2}}\right). \end{cases} \tag{2.1}$$

In view of Ciarlet [6], we have

$$\|\mathcal{P}u - u\| + \|\mathcal{Q}u - u\| \leq C\, \|u\|_{H^{k+1}}\, h^{k+1} \tag{2.2}$$

for a positive constant C independent of both u and h.

2.4 Properties of the Itô Formula

Finally we list some properties of the stochastic calculus. If X and Y are continuous semi-martingales, then the Itô formula tells us that

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s\, dY_s + \int_0^t Y_s\, dX_s + \langle X, Y\rangle_t\,,$$

where $\langle X, Y\rangle$ is the quadratic covariation process of X and Y. Note that $\langle X, Y\rangle = \langle Y, X\rangle$. For any locally bounded adapted process H, we have

$$\left\langle \int_0^{\cdot} H_s\, dX_s, Y \right\rangle_t = \int_0^t H_s\, d\,\langle X, Y\rangle_s\,. \tag{2.3}$$

Moreover, if X has bounded total variation, we have

$$\langle X, Y\rangle = 0. \tag{2.4}$$

One can prove the following lemma easily by using the dominated convergence theorem and the Burkhöder–Davis–Gundy (abbreviated as BDG) inequality.

Lemma 2.1 *If* $\mathbb{E}\left[\left(\int_0^T H_s^2\,ds\right)^{\frac{1}{2}}\right] < \infty$, *then* $\left\{\int_0^t H_s\,dW_s,\ 0 \le t \le T\right\}$ *is a martingale.*

For more details on the Itô formula, the reader is referred to [27].

3 The Ultra-Weak DG Method for the Generalized Stochastic KdV Equations

3.1 The Semi-Discrete Ultra-Weak DG Method

In this subsection, we formulate the ultra-weak DG method for the generalized stochastic KdV equations. We seek an approximation u_h to the exact solution u such that for any $(\omega, t) \in \Omega \times [0, T]$, $u_h(\omega, \cdot, t)$ belongs to the finite dimensional space V_h. In order to determine the approximate solution u_h, we first note that by multiplying (1.1) with arbitrary smooth functions v and q, and integrating over I_j with $j = 1, 2, \ldots, N$, we get, after a simple formal integration by parts,

$$
\begin{aligned}
&\int_{I_j} v(x) du(\omega, x, t)\, dx \\
&= \Bigg\{ \int_{I_j} u\,(\omega, x, t)\ v_{xxx}\,(x)\ dx \\
&\quad - u_{xx}\left(\omega, x_{j+\frac{1}{2}}, t\right)\ v\left(x^-_{j+\frac{1}{2}}\right) + u_{xx}\left(\omega, x_{j-\frac{1}{2}}, t\right)\ v\left(x^+_{j-\frac{1}{2}}\right) \\
&\quad + u_x\left(\omega, x_{j+\frac{1}{2}}, t\right)\ v_x\left(x^-_{j+\frac{1}{2}}\right) - u_x\left(\omega, x_{j-\frac{1}{2}}, t\right)\ v_x\left(x^+_{j-\frac{1}{2}}\right) \\
&\quad - u\left(\omega, x_{j+\frac{1}{2}}, t\right)\ v_{xx}\left(x^-_{j+\frac{1}{2}}\right) + u\left(\omega, x_{j-\frac{1}{2}}, t\right)\ v_{xx}\left(x^+_{j-\frac{1}{2}}\right) \\
&\quad + \int_{I_j} f\,(u\,(\omega, x, t))\ v_x\,(x)\ dx \\
&\quad - f\left(u\left(\omega, x_{j+\frac{1}{2}}, t\right)\right) v\left(x^-_{j+\frac{1}{2}}\right) + f\left(u\left(\omega, x_{j-\frac{1}{2}}, t\right)\right) v\left(x^+_{j-\frac{1}{2}}\right) \Bigg\} dt \\
&\quad + \int_{I_j} g\big(\omega, x, t, u(\omega, x, t)\big)\, v(x)\, dx\, dW_t, \\
&\int_{I_j} u(\omega, x, 0)\, q(x)\, dx = \int_{I_j} u_0(x)\, q(x)\, dx.
\end{aligned}
$$

Next, we replace the smooth functions v and q with test functions v_h and q_h, respectively, in the finite element space V_h and the exact solution u with the approximation u_h. Since the functions in V_h might have discontinuities on an element interface, we must also replace the physical fluxes

$$
u\left(\omega, x_{j+\frac{1}{2}}, t\right), \quad u_x\left(\omega, x_{j+\frac{1}{2}}, t\right), \quad u_{xx}\left(\omega, x_{j+\frac{1}{2}}, t\right) \quad \text{and} \quad f\left(u\left(\omega, x_{j+\frac{1}{2}}, t\right)\right)
$$

with the numerical fluxes

$$\widehat{u}_{j+\frac{1}{2}}(\omega,t), \quad \widetilde{u}_{x,j+\frac{1}{2}}(\omega,t), \quad \check{u}_{xx,j+\frac{1}{2}}(\omega,t) \quad \text{and} \quad \widehat{f}_{j+\frac{1}{2}}(\omega,t)$$

respectively, which will be suitably chosen later. Thus, the approximate solution given by the ultra-weak DG method is defined as the solution of the following weak formulation:

$$\begin{aligned}
\int_{I_j} v_h(x) du_h(\omega,x,t)\,dx = \Bigg\{ & \int_{I_j} u_h(\omega,x,t)\,(v_h)_{xxx}(x)\,dx \\
& - \check{u}_{xx,j+\frac{1}{2}}(\omega,t)\, v_h\left(x^-_{j+\frac{1}{2}}\right) + \check{u}_{xx,j-\frac{1}{2}}(\omega,t)\, v_h\left(x^+_{j-\frac{1}{2}}\right) \\
& + \widetilde{u}_{x,j+\frac{1}{2}}(\omega,t)\,(v_h)_x\left(x^-_{j+\frac{1}{2}}\right) - \widetilde{u}_{x,j-\frac{1}{2}}(\omega,t)\,(v_h)_x\left(x^+_{j-\frac{1}{2}}\right) \\
& - \widehat{u}_{j+\frac{1}{2}}(\omega,t)\,(v_h)_{xx}\left(x^-_{j+\frac{1}{2}}\right) + \widehat{u}_{j-\frac{1}{2}}(\omega,t)\,(v_h)_{xx}\left(x^+_{j-\frac{1}{2}}\right) \\
& + \int_{I_j} f(u_h(\omega,x,t))\,(v_h)_x(x)\,dx \\
& - \widehat{f}_{j+\frac{1}{2}}(\omega,t)\, v_h\left(x^-_{j+\frac{1}{2}}\right) + \widehat{f}_{j-\frac{1}{2}}(\omega,t)\, v_h\left(x^+_{j-\frac{1}{2}}\right) \Bigg\} dt \\
& + \int_{I_j} g\big(\omega,x,t,u_h(\omega,x,t)\big)\, v_h(x)\,dx\,dW_t, \\
\int_{I_j} u_h(\omega,x,0)\,q_h(x)\,dx = & \int_{I_j} u_0(x)\,q_h(x)\,dx.
\end{aligned} \tag{3.1}$$

It only remains to choose suitable numerical fluxes. For $j = 0, 1, \ldots, N$, we choose

$$\widehat{f}_{j+\frac{1}{2}}(\omega,t) := \widehat{f}\left(u_h\left(\omega, x^-_{j+\frac{1}{2}}, t\right), u_h\left(\omega, x^+_{j+\frac{1}{2}}, t\right)\right),$$

where the numerical flux $\widehat{f}(\cdot,\cdot)$ is a monotone flux as described in Sect. 2.2. We also choose the other numerical fluxes as

$$\widetilde{u}_{x,j+\frac{1}{2}}(\omega,t) := (u_h)_x\left(\omega, x^+_{j+\frac{1}{2}}, t\right), \tag{3.2}$$

and

$$\widehat{u}_{j+\frac{1}{2}}(\omega,t) := u_h\left(\omega, x^-_{j+\frac{1}{2}}, t\right), \quad \check{u}_{xx,j+\frac{1}{2}}(\omega,t) := (u_h)_{xx}\left(\omega, x^+_{j+\frac{1}{2}}, t\right). \tag{3.3}$$

Note that, by periodicity, we have

$$\widehat{u}_{\frac{1}{2}} = \widehat{u}_{N+\frac{1}{2}}, \quad \widetilde{u}_{x,N+\frac{1}{2}} = \widetilde{u}_{x,\frac{1}{2}}, \quad \check{u}_{xx,N+\frac{1}{2}} = \check{u}_{xx,\frac{1}{2}},$$

and

$$\widehat{f}_{\frac{1}{2}} = \widehat{f}_{N+\frac{1}{2}} = \widehat{f}\left(u_h\left(\omega, x^-_{N+\frac{1}{2}}, t\right), u_h\left(\omega, x^+_{\frac{1}{2}}, t\right)\right).$$

For simplicity of notation, for $j = 1, 2, \ldots, N$ and piece-wisely smooth functions u and v, we define

$$\begin{aligned} H_j(u, v) := & \int_{I_j} u(x)\, v_{xxx}(x)\, dx - u\left(x^-_{j+\frac{1}{2}}\right) v_{xx}\left(x^-_{j+\frac{1}{2}}\right) + u\left(x^-_{j-\frac{1}{2}}\right) v_{xx}\left(x^+_{j-\frac{1}{2}}\right) \\ & + u_x\left(x^+_{j+\frac{1}{2}}\right) v_x\left(x^-_{j+\frac{1}{2}}\right) - u_x\left(x^+_{j-\frac{1}{2}}\right) v_x\left(x^+_{j-\frac{1}{2}}\right) \\ & - u_{xx}\left(x^+_{j+\frac{1}{2}}\right) v\left(x^-_{j+\frac{1}{2}}\right) + u_{xx}\left(x^+_{j-\frac{1}{2}}\right) v\left(x^+_{j-\frac{1}{2}}\right), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} H^f_j(u, v) := & \int_{I_j} f(u)\, v_x dx - \widehat{f}\left(u\left(x^-_{j+\frac{1}{2}}\right), u\left(x^+_{j+\frac{1}{2}}\right)\right) v\left(x^-_{j+\frac{1}{2}}\right) \\ & + \widehat{f}\left(u\left(x^-_{j-\frac{1}{2}}\right), u\left(x^+_{j-\frac{1}{2}}\right)\right) v\left(x^+_{j-\frac{1}{2}}\right) \end{aligned} \tag{3.5}$$

with the numerical flux $\widehat{f}(\cdot, \cdot)$ being defined in Sect. 2.2. Then the approximate scheme (3.1) now reads

$$\begin{aligned} \int_{I_j} v_h(x) du_h(\omega, x, t)\, dx = & \left[H_j\big(u_h(\omega, \cdot, t), v_h\big) + H^f_j\big(u_h(\omega, \cdot, t), v_h\big) \right] dt \\ & + \int_{I_j} g\big(\omega, x, t, u_h(\omega, x, t)\big)\, v_h(x)\, dx\, dW_t. \end{aligned} \tag{3.6}$$

Remark 3.1 We could also define the numerical flux (3.3) in an alternative way as follows:

$$\widehat{u}_{j+\frac{1}{2}}(\omega, t) := u_h\left(\omega, x^+_{j+\frac{1}{2}}, t\right), \quad \check{u}_{xx, j+\frac{1}{2}}(\omega, t) := (u_h)_{xx}\left(\omega, x^-_{j+\frac{1}{2}}, t\right).$$

It is crucial that we take the flux $\widetilde{u}_x$ as in (3.2) and $\widehat{u}$, $\check{u}_{xx}$ from the opposite directions.

3.2 The Stochastic Ordinary Differential Equation Derived from the Spatial Discretization

The ultra-weak DG method as a spatial discretization, transfers the primal problem into a system of ordinary stochastic differential equations, which will be specified in this subsection. For $x \in I_j$ with $j = 1, 2, \ldots, N$, the numerical solution should have the form

$$u_h(\omega, x, t) = \sum_{l=0}^{k} \mathbf{u}_{l,j}(\omega, t) \varphi^j_l(x),$$

where $\{\varphi^j_l, l = 0, 1, \ldots, k\}$ is an arbitrary basis of $P^k(I_j)$.

By periodicity, we define the "ghost" coefficients as follows:

$$\mathbf{u}_{l,0} = \mathbf{u}_{l,N}, \quad \mathbf{u}_{l,N+1} = \mathbf{u}_{l,1}.$$

Our aim is to solve (3.1) to get the coefficients $\mathbf{u}(\omega, t) = [\mathbf{u}_{l,j}(\omega, t)]_{l \in \{0,\ldots,k\}, j \in \{0,\ldots,N+1\}}$. For $j = 1, 2, \ldots, N$, by taking $v_h := \varphi^j_m$ for $m = 0, 1, \ldots, k$ in equality (3.1), we have

$$
\begin{aligned}
&\sum_{n=0}^{k}\left(\int_{I_j}\varphi_m^j(x)\varphi_n^j(x)\,dx\right)d\mathbf{u}_{n,j}(\omega,t)\\
&=\Bigg\{\int_{I_j}\sum_{n=0}^{k}\mathbf{u}_{n,j}(\omega,t)\varphi_n^j(x)\left(\varphi_m^j\right)_{xxx}(x)\,dx\\
&\quad-\sum_{n=0}^{k}\Big[\mathbf{u}_{n,j+1}(\omega,t)\left(\varphi_n^{j+1}\right)_{xx}\left(x_{j+\frac12}\right)\varphi_m^j\left(x_{j+\frac12}\right)\\
&\quad-\mathbf{u}_{n,j}(\omega,t)\left(\varphi_n^{j}\right)_{xx}\left(x_{j-\frac12}\right)\varphi_m^j\left(x_{j-\frac12}\right)\Big]\\
&\quad+\sum_{n=0}^{k}\Big[\mathbf{u}_{n,j+1}(\omega,t)\left(\varphi_n^{j+1}\right)_{x}\left(x_{j+\frac12}\right)\left(\varphi_m^j\right)_x\left(x_{j+\frac12}\right)\\
&\quad-\mathbf{u}_{n,j}(\omega,t)\left(\varphi_n^{j}\right)_{x}\left(x_{j-\frac12}\right)\left(\varphi_m^j\right)_x\left(x_{j-\frac12}\right)\Big]\\
&\quad-\sum_{n=0}^{k}\Big[\mathbf{u}_{n,j}(\omega,t)\varphi_n^{j}\left(x_{j+\frac12}\right)\left(\varphi_m^j\right)_{xx}\left(x_{j+\frac12}\right)\\
&\quad-\mathbf{u}_{n,j-1}(\omega,t)\varphi_n^{j-1}\left(x_{j-\frac12}\right)\left(\varphi_m^j\right)_{xx}\left(x_{j-\frac12}\right)\Big]\\
&\quad+\int_{I_j}f\left(\sum_{n=0}^{k}\mathbf{u}_{n,j}(\omega,t)\varphi_n^j(x)\right)\varphi_{mx}^j(x)\,dx\\
&\quad-\widehat{f}\left(\sum_{n=0}^{k}\mathbf{u}_{n,j}(\omega,t)\varphi_n^j\left(x_{j+\frac12}\right),\sum_{n=0}^{k}\mathbf{u}_{n,j+1}(\omega,t)\varphi_n^{j+1}\left(x_{j+\frac12}\right)\right)\varphi_m^j\left(x_{j+\frac12}\right)\\
&\quad+\widehat{f}\left(\sum_{n=0}^{k}\mathbf{u}_{n,j-1}(\omega,t)\varphi_n^{j-1}\left(x_{j-\frac12}\right),\sum_{n=0}^{k}\mathbf{u}_{n,j}(\omega,t)\varphi_n^{j}\left(x_{j-\frac12}\right)\right)\varphi_m^j\left(x_{j-\frac12}\right)\Bigg\}dt\\
&\quad+\int_{I_j}g\left(\omega,x,t,\sum_{n=0}^{k}\mathbf{u}_{n,j}(\omega,t)\varphi_n^j(x)\right)\varphi_m^j(x)\,dx\,dW_t.
\end{aligned}
$$

The mass matrix $A^j := [A^j_{nm}]$ with

$$A^j_{nm} := \int_{I_j}\varphi_n^j(x)\varphi_m^j(x)\,dx$$

is invertible, and its inverse is denoted by $A^{j,-1}$.

Then we obtain the following SDE of $\mathbf{u}$:

$$d\mathbf{u}(t) = F\big(\mathbf{u}(t)\big)\,dt + G\big(\cdot,t,\mathbf{u}(t)\big)\,dW_t, \tag{3.7}$$

where for $j = 1, 2, \ldots, N$ and $l = 0, 1, \ldots, k$,

$$
\begin{aligned}
F_{l,j}(\mathbf{u}) &:= \int_{I_j}\sum_{n=0}^{k}\mathbf{u}_{n,j}\varphi_n^j(x)\sum_{m=0}^{k}A_{lm}^{j,-1}\left(\varphi_m^j\right)_{xxx}(x)\,dx\\
&\quad-\sum_{m=0}^{k}A_{lm}^{j,-1}\sum_{n=0}^{k}\Big[\mathbf{u}_{n,j+1}\left(\varphi_n^{j+1}\right)_{xx}\left(x_{j+\frac12}\right)\varphi_m^j\left(x_{j+\frac12}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \mathbf{u}_{n,j} \left(\varphi_n^j\right)_{xx} \left(x_{j-\frac{1}{2}}\right) \varphi_m^j \left(x_{j-\frac{1}{2}}\right)\Big] \\
&+ \sum_{m=0}^{k} A_{lm}^{j,-1} \sum_{n=0}^{k} \Big[\mathbf{u}_{n,j+1} \left(\varphi_n^{j+1}\right)_x \left(x_{j+\frac{1}{2}}\right) \left(\varphi_m^j\right)_x \left(x_{j+\frac{1}{2}}\right) \\
&- \mathbf{u}_{n,j} \left(\varphi_n^j\right)_x \left(x_{j-\frac{1}{2}}\right) \left(\varphi_m^j\right)_x \left(x_{j-\frac{1}{2}}\right)\Big] \\
&- \sum_{m=0}^{k} A_{lm}^{j,-1} \sum_{n=0}^{k} \Big[\mathbf{u}_{n,j} \varphi_n^j \left(x_{j+\frac{1}{2}}\right) \left(\varphi_m^j\right)_{xx} \left(x_{j+\frac{1}{2}}\right) \\
&- \mathbf{u}_{n,j-1} \varphi_n^{j-1} \left(x_{j-\frac{1}{2}}\right) \left(\varphi_m^j\right)_{xx} \left(x_{j-\frac{1}{2}}\right)\Big] \\
&+ \int_{I_j} f\left(\sum_{n=0}^{k} \mathbf{u}_{n,j} \varphi_n^j(x)\right) \sum_{m=0}^{k} A_{lm}^{j,-1} \varphi_{mx}^j(x)\, dx \\
&- \widehat{f}\left(\sum_{n=0}^{k} \mathbf{u}_{n,j} \varphi_n^j \left(x_{j+\frac{1}{2}}\right), \sum_{n=0}^{k} \mathbf{u}_{n,j+1} \varphi_n^{j+1} \left(x_{j+\frac{1}{2}}\right)\right) \sum_{m=0}^{k} A_{lm}^{j,-1} \varphi_m^j \left(x_{j+\frac{1}{2}}\right) \\
&+ \widehat{f}\left(\sum_{n=0}^{k} \mathbf{u}_{n,j-1} \varphi_n^{j-1} \left(x_{j-\frac{1}{2}}\right), \sum_{n=0}^{k} \mathbf{u}_{n,j} \varphi_n^j \left(x_{j-\frac{1}{2}}\right)\right) \sum_{m=0}^{k} A_{lm}^{j,-1} \varphi_m^j \left(x_{j-\frac{1}{2}}\right)
\end{aligned}
$$

and

$$
G_{l,j}(\omega, t, \mathbf{u}) := \int_{I_j} g\left(\omega, x, t, \sum_{n=0}^{k} \mathbf{u}_{n,j} \varphi_n^j(x)\right) \sum_{m=0}^{k} A_{lm}^{j,-1} \varphi_m^j(x)\, dx
$$

with periodic settings $F_{l,0} = F_{l,N}$, $F_{l,N+1} = F_{l,1}$, $G_{l,0} = G_{l,N}$, and $G_{l,N+1} = G_{l,1}$.

Lemma 3.1 *Let Assumption (H2) hold. Then for any $N \in \mathbb{N}_+$, F and G are locally Lipschitz continuous in the variable $\mathbf{u}$, i.e., for any $M \in \mathbb{N}_+$, there exists a positive constant $L_N(M)$ such that, for all $(\omega, t) \in \Omega \times [0, T]$ and all $\mathbf{u}, \mathbf{u}' \in \mathbb{R}^{(k+1)\times(N+2)}$ with $|\mathbf{u}| \vee |\mathbf{u}'| \le M$,*

$$
\left|F(\mathbf{u}) - F\left(\mathbf{u}'\right)\right| \vee \left|G(\omega, t, \mathbf{u}) - G\left(\omega, t, \mathbf{u}'\right)\right| \le L_N(M) \left|\mathbf{u} - \mathbf{u}'\right|,
$$

where the constant $L_N(M)$ may depend on N.

Proof We only show the locally Lipschitz continuity of G for fixed $N \in \mathbb{N}$, and that of F can be proved in a similar way. Note that for any $l = 0, 1, \ldots, k$, $j = 1, 2, \ldots, N$, $\mathbf{u}, \mathbf{u}' \in \mathbb{R}^{(k+1)\times(N+2)}$ with $|\mathbf{u}| \vee |\mathbf{u}'| \le M$,

$$
\begin{aligned}
&\left|G_{l,j}(\omega, t, \mathbf{u}) - G_{l,j}(\omega, t, \mathbf{u}')\right| \\
&= \Bigg|\int_{I_j} \left[g\left(\omega, x, t, \sum_{n=0}^{k} \mathbf{u}_{n,j} \varphi_n^j(x)\right) - g\left(\omega, x, t, \sum_{n=0}^{k} \mathbf{u}'_{n,j} \varphi_n^j(x)\right)\right] \\
&\quad \times \sum_{m=0}^{k} A_{lm}^{j,-1} \varphi_m^j(x)\, dx\Bigg| \\
&\le C_N(M) \sum_{n=0}^{k} \int_{I_j} \left|\varphi_n^j(x)\right| \sum_{m=0}^{k} \left|\varphi_m^j(x)\right|\, dx\, \left\|A^{j,-1}\right\|_\infty \left|\mathbf{u}_{n,j} - \mathbf{u}'_{n,j}\right|
\end{aligned}
$$

$$\leq C_N(M) \sum_{n=0}^{k} \left| \mathbf{u}_{n,j} - \mathbf{u}'_{n,j} \right| \leq C_N(M) \left(\sum_{n=0}^{k} \left| \mathbf{u}_{n,j} - \mathbf{u}'_{n,j} \right|^2 \right)^{\frac{1}{2}},$$

where $C_N(M)$ is a constant depending on N and M, and possibly changes from line to line. It leads to that

$$\left|G(\omega, t, \mathbf{u}) - G\left(\omega, t, \mathbf{u}'\right)\right|^2 = \sum_{l=0}^{k} \sum_{j=0}^{N+1} \left|G_{l,j}(\omega, t, \mathbf{u}) - G_{l,j}(\omega, t, \mathbf{u}')\right|^2$$

$$\leq \sum_{l=0}^{k} \sum_{j=0}^{N+1} C_N(M)^2 \sum_{n=0}^{k} \left| \mathbf{u}_{n,j} - \mathbf{u}'_{n,j} \right|^2 = (k+1) C_N(M)^2 \left|\mathbf{u} - \mathbf{u}'\right|^2.$$

Thus for any $N, M \in \mathbb{N}_+$, there exists a constant $L_N(M)$ such that, for all $(\omega, t) \in \Omega \times [0, T]$ and all $\mathbf{u}, \mathbf{u}' \in \mathbb{R}^{(k+1)\times(N+2)}$ with $|\mathbf{u}| \vee |\mathbf{u}'| \leq M$,

$$\left|G(\omega, t, \mathbf{u}) - G\left(\omega, t, \mathbf{u}'\right)\right| \leq L_N(M) \left|\mathbf{u} - \mathbf{u}'\right|.$$

The proof is complete. □

Similar to the proof of Lemma 3.1, we could obtain that the coefficients of SDE (3.7) satisfy the linearly growing condition.

Lemma 3.2 *Let Assumption (H3) hold. Then for any $N \in \mathbb{N}_+$, F and G are linearly growing in the variable $\mathbf{u}$, i.e., there exists a positive constant C_N such that, for all $(\omega, t) \in \Omega \times [0, T]$ and all $\mathbf{u} \in \mathbb{R}^{(k+1)\times(N+2)}$,*

$$|F(\mathbf{u})| \vee |G(\omega, t, \mathbf{u})| \leq C_N (1 + |\mathbf{u}|),$$

where the constant C_N may depend on N.

By (3.1), the initial condition of the SDE (3.7) is determined by u_0 as follows:

$$\mathbf{u}_{l,j}(\omega, 0) := \sum_{m=0}^{k} A_{lm}^{j,-1} \int_{I_j} u_0(x) \varphi_m^j(x)\, dx. \tag{3.8}$$

In the assumption (H1), u_0 is assumed to be a deterministic function. Then we know that $\mathbf{u}(0)$ is a deterministic matrix, which is $L^p(\Omega)$-integrable for any $p \geq 1$. According to the classical results of stochastic differential equations (see Mao [24]), if the initial value of the SDE is $L^p(\Omega)$-integrable and the coefficients of the SDE are locally Lipschitz continuous and linearly growing, then the considered SDE admits a unique L^p-solution. Thus, for any fixed $N \in \mathbb{N}_+$, SDE (3.7) has a unique solution $\{\mathbf{u}(t)\}_{0\leq t\leq T}$ such that for any $p \geq 1$,

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |\mathbf{u}(t)|^p\right] < \infty. \tag{3.9}$$

4 Stability Analysis for the Fully Nonlinear Equations

We have known that the approximating Eq. (3.1) has a unique solution $u_h \in V_h$ for any fixed $N \in \mathbb{N}_+$. Next we give the stability result for the numerical solutions.

Theorem 4.1 *Suppose that the assumptions (H1)–(H3) are satisfied. Then there exists a constant $C > 0$ which is independent of h, such that*

$$\sup_{0\le t\le T} \mathbb{E}\left[\|u_h(\cdot,t)\|^2\right] \le C\left(1+\|u_h(\cdot,0)\|^2\right),$$

where the constant C may depend on the terminal time T.

Proof For any $N \in \mathbb{N}_+$ and $(\omega, t) \in \Omega \times [0, T]$, by setting $v_h = u_h(\omega, \cdot, t)$ in (3.6) we have

$$\begin{aligned}
&\int_{I_j} u_h(\omega,x,t)du_h(\omega,x,t)\,dx \\
&= \left[H_j\big(u_h(\omega,\cdot,t),u_h(\omega,\cdot,t)\big)+H_j^f\big(u_h(\omega,\cdot,t),u_h(\omega,\cdot,t)\big)\right]dt \\
&\quad + \int_{I_j} g\big(\omega,x,t,u_h(\omega,x,t)\big)\,u_h(\omega,x,t)\,dx\,dW_t,
\end{aligned} \tag{4.1}$$

where the functionals H_j and H_j^f are defined by (3.4) and (3.5), respectively.

According to the Itô formula, we have for any $x \in [0, 2\pi]$,

$$|u_h(x,t)|^2 = |u_h(x,0)|^2 + 2\int_0^t u_h(x,s)\,du_h(x,s) + \langle u_h(x,\cdot),u_h(x,\cdot)\rangle_t\,.$$

Thus, after summarizing on j from 1 to N in (4.1), integrating in time from 0 to t and taking expectation we have

$$\mathbb{E}\left[\|u_h(\cdot,t)\|^2\right] = \|u_h(\cdot,0)\|^2 + \mathcal{T}_1(t)+\mathcal{T}_2(t)+\mathcal{T}_3(t)+\mathcal{T}_4(t),$$

where

$$\begin{aligned}
\mathcal{T}_1(t) &= \mathbb{E}\left[\int_0^{2\pi} \langle u_h(x,\cdot),u_h(x,\cdot)\rangle_t\ dx\right], \\
\mathcal{T}_2(t) &= 2\mathbb{E}\left[\int_0^t\int_0^{2\pi} g\big(x,s,u_h(x,s)\big)\,u_h(x,s)\,dx\,dW_s\right], \\
\mathcal{T}_3(t) &= 2\mathbb{E}\left[\int_0^t\sum_{j=1}^N H_j\big(u_h\left(\omega,\cdot,s\right),u_h\left(\omega,\cdot,s\right)\big)\,ds\right],
\end{aligned}$$

and

$$\mathcal{T}_4(t) = 2\mathbb{E}\left[\int_0^t\sum_{j=1}^N H_j^f\big(u_h\left(\omega,\cdot,s\right),u_h\left(\omega,\cdot,s\right)\big)\,ds\right].$$

Terms $\mathcal{T}_i(t)$ for $i = 1, \ldots, 4$ are estimated as follows.

- Estimate of $\mathcal{T}_1(t)$.

Compared with the deterministic case, the quadratic variation term is an essential additional term. The approximating solution u_h is given by a weak formulation (3.6) and is not easy to derive an explicit representation. Thus it is difficult to directly estimate the quadratic variation of u_h. However, we could use Fubini theorem and stochastic calculus to estimate the spatial integral of the quadratic variation. In view of (3.6), we have for any $r_h \in V_h$,

$$
\begin{aligned}
&\int_{I_j} r_h(x)u_h(x,t)\,dx \\
&= \int_{I_j} r_h(x)u_0(x)\,dx + \int_0^t \left[H_j\big(u_h(\omega,\cdot,s), r_h\big) + H_j^f\big(u_h(\omega,\cdot,s), r_h\big)\right] ds \\
&\quad + \int_0^t \int_{I_j} g\big(x,s,u_h(x,s)\big)\, r_h(x)\,dx\,dW_s.
\end{aligned}
$$

Thus by (2.4), for any continuous semimartingale Y, we obtain

$$
\begin{aligned}
\int_{I_j} r_h(x)\,\langle u_h(x,\cdot), Y\rangle_t \; dx &= \left\langle \int_{I_j} r_h(x)u_h(x,\cdot)\,dx, Y \right\rangle_t \\
&= \left\langle \int_0^{\cdot} \int_{I_j} g\big(x,s,u_h(x,s)\big)\, r_h(x)\,dx\,dW_s, Y \right\rangle_t .
\end{aligned} \tag{4.2}
$$

It turns out that

$$
\begin{aligned}
&\int_{I_j} \langle u_h(x,\cdot), u_h(x,\cdot)\rangle_t \; dx = \int_{I_j} \left\langle u_h(x,\cdot), \sum_{l=0}^{k} \mathbf{u}_{l,j}(\cdot)\varphi_l^j(x) \right\rangle_t \; dx \\
&= \sum_{l=0}^{k} \int_{I_j} \varphi_l^j(x) \left\langle u_h(x,\cdot), \mathbf{u}_{l,j}(\cdot)\right\rangle_t dx \\
&= \sum_{l=0}^{k} \left\langle \int_0^{\cdot} \int_{I_j} g\big(x,s,u_h(x,s)\big)\varphi_l^j(x) dx dW_s, \mathbf{u}_{l,j}(\cdot) \right\rangle_t .
\end{aligned}
$$

According to (2.3) and the properties of the L^2 projection, we have

$$
\begin{aligned}
&\int_{I_j} \langle u_h(x,\cdot), u_h(x,\cdot)\rangle_t \; dx \\
&\quad = \sum_{l=0}^{k} \int_0^t \int_{I_j} g\big(x,s,u_h(x,s)\big)\, \varphi_l^j(x)\,dx\,d\left\langle W, \mathbf{u}_{l,j}(\cdot)\right\rangle_s \\
&\quad = \int_{I_j} \int_0^t \sum_{l=0}^{k} \mathcal{P}\left[g\big(\cdot,s,u_h(\cdot,s)\big)\right](x)\, \varphi_l^j(x)\, d\left\langle W, \mathbf{u}_{l,j}(\cdot)\right\rangle_s \; dx \\
&\quad = \int_{I_j} \int_0^t \mathcal{P}\left[g\big(\cdot,s,u_h(\cdot,s)\big)\right](x)\, d\left\langle W, \sum_{l=0}^{k} \mathbf{u}_{l,j}(\cdot)\varphi_l^j(x) \right\rangle_s \; dx \\
&\quad = \int_{I_j} \left\langle \int_0^{\cdot} \mathcal{P}\left[g\big(\cdot,s,u_h(\cdot,s)\big)\right](x)\, dW_s, u_h(x,\cdot) \right\rangle_t \; dx.
\end{aligned}
$$

Since $\mathcal{P}\left[g\big(\cdot,s,u_h(\cdot,s)\big)\right] \in V_h$ for any $(\omega,s) \in \Omega \times [0,T]$, we have

$$
\mathcal{P}\left[g\big(\omega,\cdot,s,u_h(\omega,\cdot,s)\big)\right](x) = \sum_{l=0}^{k} \mathbf{g}_{l,j}(\omega,s)\, \varphi_l^j(x), \qquad x \in I_j.
$$

By (4.2), we get the spatial integral of quadratic variation of approximating solution u_h:

$$\begin{aligned}
\int_{I_j} \langle u_h(x,\cdot), u_h(x,\cdot)\rangle_t \, dx &= \int_{I_j} \left\langle \int_0^\cdot \sum_{l=0}^{k} \mathbf{g}_{l,j}(s)\,\varphi_l^j(x)\, dW_s, u_h(x,\cdot) \right\rangle_t dx \\
&= \sum_{l=0}^{k} \left\langle \int_0^\cdot \int_{I_j} g\big(x,s,u_h(x,s)\big)\varphi_l^j(x)\,dx\,dW_s, \int_0^\cdot \mathbf{g}_{l,j}(s)\,dW_s \right\rangle_t \\
&= \sum_{l=0}^{k} \int_0^t \int_{I_j} g\big(x,s,u_h(x,s)\big)\varphi_l^j(x)\,dx\,\mathbf{g}_{l,j}(s)\,d\langle W,W\rangle_s \\
&= \int_0^t \int_{I_j} g\big(x,s,u_h(x,s)\big)\mathcal{P}\left[g\big(\cdot,s,u_h(\cdot,s)\big)\right](x)\,dx\,ds. \qquad (4.3)
\end{aligned}$$

After summarizing over j from 1 to N, by Cauchy–Schwartz's inequality we have

$$\int_0^{2\pi} \langle u_h(x,\cdot), u_h(x,\cdot)\rangle_t \, dx \le \int_0^t \int_0^{2\pi} \left|g\big(x,s,u_h(x,s)\big)\right|^2 dx\,ds.$$

According to (H3), after taking expectation, we have

$$\begin{aligned}
\mathcal{T}_1(t) = \mathbb{E}\left[\int_0^{2\pi} \langle u_h(x,\cdot), u_h(x,\cdot)\rangle_t \, dx\right] &\le \mathbb{E}\left[\int_0^t \int_0^{2\pi} \left|g\big(x,s,u_h(x,s)\big)\right|^2 dx\,ds\right] \\
&\le C + C\int_0^t \mathbb{E}\left[\|u_h(\cdot,s)\|^2\right] ds.
\end{aligned}$$

- Estimate of $\mathcal{T}_2(t)$.

From (3.9), we have for any fixed $N \in \mathbb{N}_+$,

$$\mathbb{E}\left[\sup_{0\le s\le T} \|u_h(\cdot,s)\|^2\right] < \infty. \qquad (4.4)$$

Thus by (H3) and Cauchy–Schwartz's inequality we know that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T \left|\int_0^{2\pi} g\big(x,s,u_h(x,s)\big)\,u_h(x,s)\,dx\right|^2 ds\right)^{\frac{1}{2}}\right] \\
&\le \mathbb{E}\left[\left(\int_0^T \|u_h(\cdot,s)\|^2 \int_0^{2\pi} \left|g\big(x,s,u_h(x,s)\big)\right|^2 dxds\right)^{\frac{1}{2}}\right] \\
&\le C\mathbb{E}\left[\sup_{0\le s\le T} \|u_h(\cdot,s)\| \left(\int_0^T \int_0^{2\pi} \left(1+|u_h(x,s)|^2\right) dxds\right)^{\frac{1}{2}}\right] \\
&\le C\left(\mathbb{E}\left[\sup_{0\le s\le T} \|u_h(\cdot,s)\|^2\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\int_0^T \left(1+\|u_h(\cdot,s)\|^2\right) ds\right]\right)^{\frac{1}{2}} < \infty.
\end{aligned}$$

According to Lemma 2.1, the process

$$\left\{\int_0^t \int_0^{2\pi} g\big(x,s,u_h(x,s)\big)\,u_h(x,s)\,dx\,dW_s, \quad 0\le t\le T\right\}$$

is a martingale. It turns out that

$$\mathcal{T}_2(t) = 2\mathbb{E}\left[\int_0^t \int_0^{2\pi} g\big(x, s, u_h(x, s)\big)\, u_h(x, s)\, dx\, dW_s\right] = 0.$$

- Estimate of $\mathcal{T}_3(t)$.

For any $u \in V_h$, we have

$$\begin{aligned}
H_j(u, u) = &\int_{I_j} u\,(x)\; u_{xxx}(x)\, dx - u\left(x^-_{j+\frac{1}{2}}\right) u_{xx}\left(x^-_{j+\frac{1}{2}}\right) + u\left(x^-_{j-\frac{1}{2}}\right) u_{xx}\left(x^+_{j-\frac{1}{2}}\right)\\
&+ u_x\left(x^+_{j+\frac{1}{2}}\right) u_x\left(x^-_{j+\frac{1}{2}}\right) - u_x\left(x^+_{j-\frac{1}{2}}\right) u_x\left(x^+_{j-\frac{1}{2}}\right)\\
&- u_{xx}\left(x^+_{j+\frac{1}{2}}\right) u\left(x^-_{j+\frac{1}{2}}\right) + u_{xx}\left(x^+_{j-\frac{1}{2}}\right) u\left(x^+_{j-\frac{1}{2}}\right)\\
= &- \int_{I_j} u_x\,(x)\; u_{xx}(x)\, dx + u\left(x^-_{j+\frac{1}{2}}\right) u_{xx}\left(x^-_{j+\frac{1}{2}}\right) - u\left(x^+_{j-\frac{1}{2}}\right) u_{xx}\left(x^+_{j-\frac{1}{2}}\right)\\
&- u\left(x^-_{j+\frac{1}{2}}\right) u_{xx}\left(x^-_{j+\frac{1}{2}}\right) + u\left(x^-_{j-\frac{1}{2}}\right) u_{xx}\left(x^+_{j-\frac{1}{2}}\right)\\
&+ u_x\left(x^+_{j+\frac{1}{2}}\right) u_x\left(x^-_{j+\frac{1}{2}}\right) - u_x\left(x^+_{j-\frac{1}{2}}\right) u_x\left(x^+_{j-\frac{1}{2}}\right)\\
&- u_{xx}\left(x^+_{j+\frac{1}{2}}\right) u\left(x^-_{j+\frac{1}{2}}\right) + u_{xx}\left(x^+_{j-\frac{1}{2}}\right) u\left(x^+_{j-\frac{1}{2}}\right)\\
= &- \frac{1}{2}\left|u_x\left(x^-_{j+\frac{1}{2}}\right)\right|^2 + \frac{1}{2}\left|u_x\left(x^+_{j-\frac{1}{2}}\right)\right|^2 + u\left(x^-_{j-\frac{1}{2}}\right) u_{xx}\left(x^+_{j-\frac{1}{2}}\right)\\
&+ u_x\left(x^+_{j+\frac{1}{2}}\right) u_x\left(x^-_{j+\frac{1}{2}}\right) - u_x\left(x^+_{j-\frac{1}{2}}\right) u_x\left(x^+_{j-\frac{1}{2}}\right) - u_{xx}\left(x^+_{j+\frac{1}{2}}\right) u\left(x^-_{j+\frac{1}{2}}\right).
\end{aligned}$$

By periodicity, we get

$$\begin{aligned}
\sum_{j=1}^N H_j(u, u) = &\sum_{j=1}^N \Bigg[- \frac{1}{2}\left|u_x\left(x^-_{j+\frac{1}{2}}\right)\right|^2 + \frac{1}{2}\left|u_x\left(x^+_{j+\frac{1}{2}}\right)\right|^2 + u\left(x^-_{j+\frac{1}{2}}\right) u_{xx}\left(x^+_{j+\frac{1}{2}}\right)\\
&+ u_x\left(x^+_{j+\frac{1}{2}}\right) u_x\left(x^-_{j+\frac{1}{2}}\right) - \left|u_x\left(x^+_{j+\frac{1}{2}}\right)\right|^2 - u_{xx}\left(x^+_{j+\frac{1}{2}}\right) u\left(x^-_{j+\frac{1}{2}}\right)\Bigg]\\
= &\sum_{j=1}^N \Bigg[- \frac{1}{2}\left|u_x\left(x^-_{j+\frac{1}{2}}\right)\right|^2 - \frac{1}{2}\left|u_x\left(x^+_{j+\frac{1}{2}}\right)\right|^2 + u_x\left(x^+_{j+\frac{1}{2}}\right) u_x\left(x^-_{j+\frac{1}{2}}\right)\Bigg]\\
= &- \frac{1}{2}\sum_{j=1}^N \left|u_x\left(x^-_{j+\frac{1}{2}}\right) - u_x\left(x^+_{j+\frac{1}{2}}\right)\right|^2.
\end{aligned}$$

Thus for any $u \in V_h$

$$\sum_{j=1}^N H_j(u, u) \le 0. \tag{4.5}$$

It gives that

$$\mathcal{T}_3(t) = 2\mathbb{E}\left[\int_0^t \sum_{j=1}^N H_j\big(u_h(\omega,\cdot,s), u_h(\omega,\cdot,s)\big)\, ds\right] \le 0.$$

- Estimate of $\mathcal{T}_4(t)$.

For any $u \in V_h$, we have

$$\begin{aligned}
\sum_{j=1}^N H_j^f\big(u,u\big) &= \sum_{j=1}^N \left[\int_{I_j} f(u)\, u_x dx - \widehat{f}\left(u^-_{j+\frac12}, u^+_{j+\frac12}\right) u^-_{j+\frac12} + \widehat{f}\left(u^-_{j-\frac12}, u^+_{j-\frac12}\right) u^+_{j-\frac12}\right] \\
&= \sum_{j=1}^N \left[\phi\left(u^-_{j+\frac12}\right) - \phi\left(u^+_{j-\frac12}\right) - \widehat{f}_{j+\frac12} u^-_{j+\frac12} + \widehat{f}_{j-\frac12} u^+_{j-\frac12}\right] \\
&= \sum_{j=1}^N \left(\widehat{F}_{j+\frac12} - \widehat{F}_{j-\frac12} + \Theta_{j-\frac12}\right),
\end{aligned}$$

where

$$\begin{aligned}
\phi(u) &= \int^u f(a)\, da, \\
\widehat{F}_{j+\frac12} &= \left(\phi(u^-) - \widehat{f}\cdot u^-\right)_{j+\frac12}, \\
\Theta_{j-\frac12} &= \left[\phi(u^-) - \phi(u^+) + \widehat{f}\cdot\left(u^+ - u^-\right)\right]_{j-\frac12}.
\end{aligned}$$

By periodicity, we have

$$\sum_{j=1}^N \left(\widehat{F}_{j+\frac12} - \widehat{F}_{j-\frac12}\right) = 0.$$

Note that

$$\begin{aligned}
\Theta &= \phi(u^-) - \phi(u^+) + \widehat{f}(u^-, u^+)\left(u^+ - u^-\right) \\
&= \phi'(\xi)(u^+ - u^-) + \widehat{f}(u^-, u^+)\left(u^+ - u^-\right) \\
&= \left(\widehat{f}(u^-, u^+) - \widehat{f}(\xi,\xi)\right)\left(u^+ - u^-\right) \\
&= \left(\widehat{f}(u^-, u^+) - \widehat{f}(u^-,\xi) + \widehat{f}(u^-,\xi) - \widehat{f}(\xi,\xi)\right)\left(u^+ - u^-\right) \le 0,
\end{aligned}$$

where ξ is a real number between u^- and u^+. Thus for any $u \in V_h$

$$\sum_{j=1}^N H_j^f\big(u,u\big) \le 0. \tag{4.6}$$

It turns out that

$$\mathcal{T}_4(t) = 2\mathbb{E}\left[\int_0^t \sum_{j=1}^N H_j^f\big(u_h(\omega,\cdot,s), u_h(\omega,\cdot,s)\big)\, ds\right] \le 0.$$

Then there exists a positive constant C which is independent of h, such that for any $t \in [0,T]$,

$$\mathbb{E}\left[\|u_h(\cdot,t)\|^2\right] \le \|u_h(\cdot,0)\|^2 + C + C\int_0^t \mathbb{E}\left[\|u_h(\cdot,s)\|^2\right] ds.$$

Using Gronwall's inequality, we have for any $t \in [0, T]$,

$$\mathbb{E}\left[\|u_h(\cdot, t)\|^2\right] \le \left(C + \|u_h(\cdot, 0)\|^2\right) e^{Ct}.$$

This completes the proof. □

5 Optimal Error Estimates for Semilinear Equations

In this section, we consider the convergence of numerical method for strong solutions with enough smoothness and integrability. We prove the optimal error estimates ($\mathcal{O}(h^{k+1})$) with respect to spatial $L^2(0, 2\pi)$-norm for the semilinear case that $f(u) := 0$,

$$\begin{cases} du = -u_{xxx}\, dt + g(\cdot, x, t, u)\, dW_t, & (x, t) \in [0, 2\pi] \times (0, T]; \\ u(x, 0) = u_0(x), & x \in [0, 2\pi]. \end{cases} \tag{5.1}$$

In the semilinear case, the ultra-weak DG method (3.1) can be written as follows. For any $(\omega, t) \in \Omega \times [0, T]$, find $u_h(\omega, \cdot, t) \in V_h$ such that for any $v_h \in V_h$,

$$\int_{I_j} v_h(x) du_h(\omega, x, t)\, dx = H_j\big(u_h(\omega, \cdot, t), v_h\big)dt + \int_{I_j} g\big(\omega, x, t, u_h(\omega, x, t)\big)v_h(x)dxdW_t, \tag{5.2}$$

where the bilinear functional H_j is defined by (3.4). Then, we state the error estimates of the semi-discrete ultra-weak DG scheme (5.2).

Theorem 5.1 *Suppose that $u_0 \in H^{k+1}$ with $k \ge 2$, the coefficient $g(\cdot)$ is uniformly Lipschitz continuous in u, and Eq. (5.1) has a unique strong solution $u(\cdot)$ such that*

(H4) $u(\cdot) \in L^2\left(\Omega \times [0, T]; H^{k+4}\right) \bigcap S^2\left(\Omega \times [0, T]; L^2\right) \bigcap L^\infty\left(0, T; L^2(\Omega; H^{k+1})\right)$;
(H5) $g(\cdot, u(\cdot)) \in L^2\left(\Omega \times [0, T]; H^{k+1}\right)$.

Then, there is a positive constant C which is independent of h, such that

$$\sup_{t \in [0,T]} \left(\mathbb{E}\left[\|u(\cdot, t) - u_h(\cdot, t)\|^2\right]\right)^{\frac{1}{2}} \le Ch^{k+1}, \tag{5.3}$$

where the constant C may depend on the terminal time T.

Proof Note that the scheme (5.2) is also satisfied when the numerical solution $u_h(\cdot)$ is replaced with the exact solution $u(\cdot)$: for any $(\omega, t) \in \Omega \times [0, T]$ and $v_h \in V_h$, we have

$$\int_{I_j} v_h(x) du(\omega, x, t)\, dx = H_j\big(u(\omega, \cdot, t), v_h\big)dt + \int_{I_j} g\big(\omega, x, t, u(\omega, x, t)\big)\, v_h(x)\, dx\, dW_t.$$

Define

$$e(\omega, x, t) := (u - u_h)(\omega, x, t) = (\xi - \eta)(\omega, x, t),$$

with

$$\xi(\omega, x, t) := (\mathcal{Q}u - u_h)(\omega, x, t), \quad \eta(\omega, x, t) := (\mathcal{Q}u - u)(\omega, x, t)$$

where $\mathcal{Q}$ is the projection from H^{k+1} onto V_h defined in (2.1).

Then the error equation is

$$\int_{I_j} v_h(x)\, d\, e(\omega, x, t)\, dx$$

$$= H_j\big(e(\omega,\cdot,t), v_h\big)dt + \int_{I_j} \big[g\big(\omega,x,t,u(\omega,x,t)\big) - g\big(\omega,x,t,u_h(\omega,x,t)\big)\big]\, v_h(x)dx dW_t.$$

Taking $v_h = \xi(\omega,\cdot,t)$, we have

$$\int_{I_j} \xi(x,t)d\xi(x,t)dx = \int_{I_j} \xi(x,t)d\eta(x,t)dx + \Big[H_j\big(\xi(\cdot,t),\xi(\cdot,t)\big) - H_j\big(\eta(\cdot,t),\xi(\cdot,t)\big)\Big]dt + \int_{I_j}\Big[g\big(x,t,u(x,t)\big) - g\big(x,t,u_h(x,t)\big)\Big]\xi(x,t)\,dx\,dW_t.$$

Using the Itô's formula, we have for any $x \in [0, 2\pi]$,

$$d\,|\xi(x,t)|^2 = 2\xi(x,t)\,d\xi(x,t) + d\,\langle \xi(x,\cdot),\xi(x,\cdot)\rangle_t\,.$$

Then, we have

$$\mathbb{E}\left[\|\xi(\cdot,t)\|^2\right] = \|\xi(\cdot,0)\|^2 + \mathcal{T}_1(t) + \mathcal{T}_2(t) + \mathcal{T}_3(t) + \mathcal{T}_4(t) + \mathcal{T}_5(t)$$

where

$$\mathcal{T}_1(t) := 2\mathbb{E}\left[\int_0^{2\pi}\int_0^t \xi(x,s)d\eta(x,s)\,dx\right],$$

$$\mathcal{T}_2(t) := \mathbb{E}\left[\int_0^{2\pi} \langle\xi(x,\cdot),\xi(x,\cdot)\rangle_t\;dx\right],$$

$$\mathcal{T}_3(t) := 2\mathbb{E}\left[\int_0^t \sum_{j=1}^N H_j\big(\xi(\cdot,s),\xi(\cdot,s)\big)\,ds\right],$$

$$\mathcal{T}_4(t) := -2\mathbb{E}\left[\int_0^t \sum_{j=1}^N H_j\big(\eta(\cdot,s),\xi(\cdot,s)\big)\,ds\right],$$

and

$$\mathcal{T}_5(t) := 2\mathbb{E}\left[\int_0^t\int_0^{2\pi}\left[g\left(x,s,u(x,s)\right) - g\left(x,s,u_h(x,s)\right)\right]\xi(x,s)\,dx\,dW_s\right].$$

The terms $\mathcal{T}_i(t)$ for $i = 1,\ldots,5$ are estimated as follows.

- Estimate of $\mathcal{T}_1(t)$.

In view of (5.1), we have

$$d_t(\mathcal{Q}u)(\cdot,t) = \mathcal{Q}(d_t u)(\cdot,t) = -\,\mathcal{Q}\left[u_{xxx}(\cdot,t)\right]dt + \mathcal{Q}\left[g(\cdot,t,u(\cdot,t))\right]dW_t. \quad (5.4)$$

Therefore,

$$d\eta(\cdot,t) = -(\mathcal{Q}u_{xxx} - u_{xxx})(\cdot,t)\,dt + (\mathcal{Q}-\mathcal{I})g(\cdot,t,u(\cdot,t))\,dW_t$$

with $\mathcal{I}$ being the identity operator.

It turns out that

$$\int_0^{2\pi} \xi(x,t)\,d\eta(x,t)\,dx = -\int_0^{2\pi} \xi(x,t)\left(\mathcal{Q}u_{xxx} - u_{xxx}\right)(x,t)dxdt + \int_0^{2\pi} \xi(x,t)\,(\mathcal{Q}-\mathcal{I})\left[g(\cdot,t,u(\cdot,t))\right](x)\,dx\,dW_t.$$

Since $u(\cdot) \in S^2\left(\Omega \times [0,T]; L^2\right)$, we have $\mathcal{Q}u(\cdot) \in S^2\left(\Omega \times [0,T]; L^2\right)$. By (4.4), we get

$$\mathbb{E}\left[\sup_{0\le s\le T} \|\xi(\cdot,s)\|^2\right] < \infty.$$

Thus by virtue of (H3) and Cauchy–Schwartz's inequality we know that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T \left|\int_0^{2\pi} \xi(x,s)\,(\mathcal{Q}-\mathcal{I})\big[g(\cdot,t,u(\cdot,s))\big](x)\,dx\right|^2 ds\right)^{\frac{1}{2}}\right]\\
&\le \mathbb{E}\left[\left(\int_0^T \|\xi(\cdot,s)\|^2 \int_0^{2\pi} \left|(\mathcal{Q}-\mathcal{I})\left[g(\cdot,s,u(\cdot,s))\right]\right|^2(x)\,dxds\right)^{\frac{1}{2}}\right]\\
&\le C\mathbb{E}\left[\sup_{0\le s\le T}\|\xi(\cdot,s)\|\left(\int_0^T\int_0^{2\pi}\left(1+|u(x,s)|^2\right)dxds\right)^{\frac{1}{2}}\right]\\
&\le C\left(\mathbb{E}\left[\sup_{0\le s\le T}\|\xi(\cdot,s)\|^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\int_0^T\left(1+\|u(\cdot,s)\|^2\right)ds\right]\right)^{\frac{1}{2}} < \infty. \qquad (5.5)
\end{aligned}$$

According to Lemma 2.1, we could verify that the process

$$\int_0^t\int_0^{2\pi}\xi(x,s)\,(\mathcal{Q}-\mathcal{I})\big[g(\cdot,s,u(\cdot,s))\big](x)\,dx\,dW_s, \quad 0\le t\le T$$

is a martingale. Thus according to the property of the projection (2.2), we have

$$\begin{aligned}
\mathcal{T}_1(t) &= -2\mathbb{E}\left[\int_0^t\int_0^{2\pi}\xi(x,s)\,(\mathcal{Q}u_{xxx}-u_{xxx})\,(x,s)dxds\right]\\
&\le \mathbb{E}\left[\int_0^t\left(\|\xi\,(\cdot,s)\|^2+\|(\mathcal{Q}u_{xxx}-u_{xxx})\,(\cdot,s)\|^2\right)ds\right]\\
&\le \int_0^t \mathbb{E}\,\|\xi\,(\cdot,s)\|^2\,ds + Ch^{2k+2}\mathbb{E}\left[\int_0^t\|u_{xxx}\,(\cdot,s)\|^2_{H^{k+1}}\,ds\right].
\end{aligned}$$

Since

$$u\in L^2\left(\Omega\times[0,T];H^{k+4}\right),$$

we have

$$\mathcal{T}_1(t)\le\int_0^t\mathbb{E}\,\|\xi\,(\cdot,s)\|^2\,ds + C\,h^{2k+2}.$$

- Estimate of $\mathcal{T}_2(t)$.

In view of (5.4), we have that for any $v_h\in V_h$,

$$\begin{aligned}
&\int_{I_j} v_h(x)d\mathcal{Q}u(x,t)\,dx\\
&\quad= -\int_{I_j} v_h(x)\mathcal{Q}\left[u_{xxx}(\cdot,t)\right](x)dxdt\\
&\qquad+\int_{I_j} v_h(x)\mathcal{Q}\big[g(\cdot,t,u(\cdot,t))\big](x)\,dx\,dW_t. \qquad (5.6)
\end{aligned}$$

From (5.2) and (5.6), we obtain that for any $v_h \in V_h$,

$$\int_{I_j} v_h(x) d\xi(x,t) dx = -\left\{ \int_{I_j} v_h(x)\, \mathcal{Q}\left[u_{xxx}(\cdot,t)\right](x)\, dx + H_j\big(u_h(\cdot,t), v_h\big) \right\} dt$$
$$+ \int_{I_j} v_h(x) \left\{ \mathcal{Q}\big[g(\cdot,t,u(\cdot,t)\big] \right.$$
$$\left. -g\big(\cdot,t,u_h(\cdot,t)\big) \right\}(x) dx dW_t. \tag{5.7}$$

Since $\xi(\omega,\cdot,t) \in V_h$ for any $(\omega,t) \in \Omega \times [0,T]$, then $\xi(\cdot)$ should have the form

$$\xi(\omega,x,t) = \sum_{l=0}^{k} \tilde{\xi}_{l,j}(\omega,t)\varphi_l^j(x), \quad x \in I_j.$$

Similar to (4.3), we have from (5.7) that

$$\int_{I_j} \langle \xi(x,\cdot), \xi(x,\cdot)\rangle_t\, dx = \int_0^t \int_{I_j} \Big(\mathcal{P}\left\{ \mathcal{Q}\big[g(\cdot,s,u(\cdot,s))\big] - g\big(\cdot,s,u_h(\cdot,s)\big)\right\}(x)$$
$$\times \left\{ \mathcal{Q}\big[g(\cdot,s,u(\cdot,s))\big] - g\big(\cdot,s,u_h(\cdot,s)\big)\right\}(x) \Big) dx ds$$
$$\leq \int_0^t \int_{I_j} \left| \mathcal{Q}\big[g(\cdot,s,u(\cdot,s))\big] - g\big(\cdot,s,u_h(\cdot,s)\big)\right|^2 (x)\, dx\, ds.$$

Then we get

$$\mathcal{T}_2(t) = \mathbb{E}\left[\int_0^{2\pi} \langle \xi(x,\cdot), \xi(x,\cdot)\rangle_t\, dx \right]$$
$$\leq \mathbb{E}\left[\int_0^t \int_0^{2\pi} \left| \mathcal{Q}\big[g(\cdot,s,u(\cdot,s))\big] - g\big(\cdot,s,u_h(\cdot,s)\big)\right|^2 (x)\, dx\, ds \right]$$
$$\leq 2\mathbb{E}\left[\int_0^t \int_0^{2\pi} \left| (\mathcal{Q} - \mathcal{I})\, g\big(\cdot,s,u(\cdot,s)\big)\right|^2 (x)\, dx\, ds \right]$$
$$+ 2\mathbb{E}\left[\int_0^t \int_0^{2\pi} \left| g\big(x,s,u(x,s)\big) - g\big(x,s,u_h(x,s)\big)\right|^2 dx\, ds \right].$$

According to (H5) and the property of the projection, we have

$$\mathcal{T}_2(t) \leq Ch^{2k+2}\mathbb{E}\left[\int_0^t \|g(\cdot,s,u(\cdot,s))\|_{H^{k+1}}^2\, ds \right]$$
$$+ C\mathbb{E}\int_0^t \int_0^{2\pi} \left[|\eta(x,s)|^2 + |\xi(x,s)|^2 \right] dx\, ds$$
$$\leq Ch^{2k+2} + Ch^{2k+2}\mathbb{E}\left[\int_0^t \|u(\cdot,s)\|_{H^{k+1}}^2\, ds \right] + C\mathbb{E}\left[\int_0^t \|\xi(\cdot,s)\|^2\, ds \right].$$

Since $u \in L^2\left(\Omega \times [0,T]; H^{k+4}\right) \subseteq L^2\left(\Omega \times [0,T]; H^{k+1}\right)$, we have

$$\mathcal{T}_2(t) \leq Ch^{2k+2} + C\int_0^t \mathbb{E}\left[\|\xi(\cdot,s)\|^2 \right] ds.$$

- Estimate of $\mathcal{T}_3(t)$.

According to (4.5), for any $u \in V_h$, we have

$$\sum_{j=1}^{N} H_j(u, u) \leq 0.$$

Since $\xi(\omega, \cdot, t)$ is in V_h for any $(\omega, t) \in \Omega \times [0, T]$, we get

$$\mathcal{T}_3(t) = 2\mathbb{E}\left[\int_0^t \sum_{j=1}^{N} H_j\big(\xi(\cdot, s), \xi(\cdot, s)\big)\, ds\right] \leq 0,$$

- Estimate of $\mathcal{T}_4(t)$.

By the definition of the projections $\mathcal{Q}$ (see (2.1)), we see that for any $(\omega, t) \in \Omega \times [0, T]$, $j = 1, 2, \ldots, N$,

$$\begin{cases} \displaystyle\int_{I_j} \eta(\omega, x, t)\, r(x)\, dx = 0, \quad \forall r \in P^{k-3}(I_j), \\ \eta\left(\omega, x^-_{j+\frac{1}{2}}, t\right) = 0, \\ \eta_x\left(\omega, x^+_{j-\frac{1}{2}}, t\right) = 0, \\ \eta_{xx}\left(\omega, x^+_{j-\frac{1}{2}}, t\right) = 0. \end{cases} \tag{5.8}$$

According to (3.4), we have for any $v \in V_h$,

$$\begin{aligned} H_j\left(\eta(\omega, \cdot, t), v\right) = & \int_{I_j} \eta(\omega, x, t)\; v_{xxx}(x)\, dx \\ & - \eta\left(\omega, x^-_{j+\frac{1}{2}}, t\right)\; v_{xx}\left(x^-_{j+\frac{1}{2}}\right) + \eta\left(\omega, x^-_{j-\frac{1}{2}}, t\right)\; v_{xx}\left(x^+_{j-\frac{1}{2}}\right) \\ & + \eta_x\left(\omega, x^+_{j+\frac{1}{2}}, t\right)\; v_x\left(x^-_{j+\frac{1}{2}}\right) - \eta_x\left(\omega, x^+_{j-\frac{1}{2}}, t\right)\; v_x\left(x^+_{j-\frac{1}{2}}\right) \\ & - \eta_{xx}\left(\omega, x^+_{j+\frac{1}{2}}, t\right)\; v\left(x^-_{j+\frac{1}{2}}\right) + \eta_{xx}\left(\omega, x^+_{j-\frac{1}{2}}, t\right)\; v\left(x^+_{j-\frac{1}{2}}\right) = 0. \end{aligned}$$

Since $\xi(\omega, \cdot, t) \in V_h$, we have

$$\mathcal{T}_4(t) = -\,2\mathbb{E}\left[\int_0^t \sum_{j=1}^{N} H_j\big(\eta(\cdot, s), \xi(\cdot, s)\big)\, ds\right] = 0.$$

- Estimate of $\mathcal{T}_5(t)$.

By virtue of (4.4) and $u(\cdot) \in S^2\left(\Omega \times [0, T]; L^2\right)$, similar to (5.5), we get

$$\mathbb{E}\left[\left(\int_0^T \left|\int_0^{2\pi} \left[g\left(x, s, u(x, s)\right)\right) - g\left(x, s, u_h(x, s)\right)\right] \xi(x, s)\, dx\right|^2 ds\right)^{\frac{1}{2}}\right] < \infty.$$

According to Lemma 2.1, we see that the process

$$\int_0^t \int_0^{2\pi} \left[g\left(x, s, u(x, s)\right)\right) - g\left(x, s, u_h(x, s)\right)\right] \xi(x, s)\, dx dW_s, \quad 0 \leq t \leq T$$

is a martingale. Thus,

$$\mathcal{T}_5(t) = 2\mathbb{E}\left[\int_0^t \int_0^{2\pi} \left[g\left(x, s, u(x, s)\right) - g\left(x, s, u_h(x, s)\right)\right]\xi(x, s)\, dx\, dW_s\right] = 0.$$

Concluding the above, we have

$$\mathbb{E}\left[\|\xi(\cdot, t)\|^2\right] \le \|\xi(\cdot, 0)\|^2 + Ch^{2k+2} + C\int_0^t \mathbb{E}\left[\|\xi(\cdot, s)\|^2\right] ds.$$

Since $\|\xi(\cdot, 0)\| = \|\mathcal{Q}u_0 - \mathcal{P}u_0\| \le Ch^{k+1}\|u_0\|_{H^{k+1}}$, we have from Gronwall's inequality that

$$\left(\mathbb{E}\left[\|\xi(\cdot, t)\|^2\right]\right)^{\frac{1}{2}} \le Ch^{k+1}e^{Ct}.$$

Since $u \in L^\infty\left(0, T; L^2(\Omega; H^{k+1})\right)$, we have

$$\left(\mathbb{E}\left[\|\eta(\cdot, t)\|^2\right]\right)^{\frac{1}{2}} \le C\left(\mathbb{E}\left[\|u(\cdot, t)\|^2_{H^{k+1}}\right]\right)^{\frac{1}{2}} h^{k+1} \le Ch^{k+1}.$$

It turns out that

$$\left(\mathbb{E}\left[\|u(\cdot, t) - u_h(\cdot, t)\|^2\right]\right)^{\frac{1}{2}} \le \left(\mathbb{E}\left[\|\xi(\cdot, t)\|^2\right]\right)^{\frac{1}{2}} + \left(\mathbb{E}\left[\|\eta(\cdot, t)\|^2\right]\right)^{\frac{1}{2}} \le Ce^{Ct}h^{k+1}.$$

□

Remark 5.1 It should be pointed out that the regularity condition (H4) seems to be stringent. We find no literature on the regularity of a strong solution to Eq. (5.1). However, our examples (see (7.1)–(7.3)) demonstrate that there is a sufficiently broad class of problems satisfying assumption (H4), as long as the corresponding deterministic initial values u_0 have enough regularities.

On the other hand, in practice if such regularities could not be achieved, we could consider the weak version of the scheme. We only need to assume that the coefficient $g(\cdot)$ satisfies some regularity such that Eq. (5.1) has a unique strong solution $u(\cdot)$ and the processes

$$\int_0^t \int_{I_j} g(x, s, u(x, s))\, dx\, dW_s, \quad \int_0^t \int_{I_j} g(x, s, u_h(x, s)) v_h(x)\, dx\, dW_s, \qquad 0 \le t \le T$$

are martingales. Then by taking expectation on both sides of (5.1) and (5.2), we get

$$\begin{cases} \bar{u}_t = -\bar{u}_{xxx}, & (x, t) \in [0, 2\pi] \times (0, T]; \\ \bar{u}(x, 0) = u_0(x), & x \in [0, 2\pi]. \end{cases} \tag{5.9}$$

and

$$\int_{I_j} v_h(x)\ (\bar{u}_h)_t\,(x, t)\, dx = H_j\left(\bar{u}_h(\cdot, t), v_h\right), \tag{5.10}$$

where $\bar{u} = \mathbb{E}[u]$ and $\bar{u}_h = \mathbb{E}[u_h]$. We see that (5.9) is the simple third-order deterministic PDE and (5.10) is the corresponding classical ultra-weak DG method. In this case, though we could not get the strong result (5.3), we still could obtain the weak result without (H4) and (H5)

$$\sup_{t \in [0, T]} \|\mathbb{E}[u(\cdot, t) - u_h(\cdot, t)]\| \le Ch^{k+1}.$$

Remark 5.2 In the estimation of $\mathcal{T}_4(t)$, it is essential to set $k \geq 2$ to get the error estimate. If $k < 2$, then we could not well define the projection $\mathcal{Q}$ as (2.1), which leads to that (5.8) will not hold and $\mathcal{T}_4(t)$ cannot be estimated. This is also the case for deterministic KdV equations. When $k < 2$, numerical experiments in Sect. 7 also show that our scheme is not consistent.

Remark 5.3 In the deterministic setting, the ultra-weak DG method focuses on high-order convergence of strong solution. As the stochastic counterpart, we naturally consider the high-order convergence of strong solution. As a consequence, the mean-square convergence for stochastic KdV equations is considered. Note that the mean-square convergence could also derive the weak convergence.

Remark 5.4 The solutions of the stochastic KdV equations rarely have a uniform bound with respect to the variable $\omega \in \Omega$. Thus it is difficult to use the method in Zhang and Shu [35] to get error estimates for the stochastic equation containing the nonlinear term $f(\cdot)$, which requires the uniform boundedness of the approximate solutions. But interestingly, numerical examples in Sect. 7.3 verify the optimal order $\mathcal{O}(h^{k+1})$ for nonlinear stochastic equations.

6 IMEX Time Discretization

The ultra-weak DG method incorporates the spatial discretization and reduces the primal SPDE into a system of SDEs, which needs to be coupled with a high-order time discretization. The second-order explicit methods used in [21] are stable, efficient and accurate for solving hyperbolic conservation laws. However, KdV equations contain third-order spatial derivatives. For these problems which are not convection-dominated, explicit time discretization will suffer from a stringent time-step restriction $\Delta t \sim (\Delta x)^3$ for stability. When it comes to such problems, a natural consideration to overcome the small time-step restriction is to use implicit time-marching.

Implicit schemes are thoroughly discussed in [26], motivated by long-time integration with geometry-preserving properties. These properties could well fit the need for long-time integration. Also, implicit schemes (e.g., midpoint scheme) may provide the computational reduction for numerical SDEs with a single noise.

In fact, in many applications the convection terms are often nonlinear; hence it would be desirable to treat them explicitly while using implicit time discretization only for the third-order linear term in the KdV equations. Such time discretizations are called implicit–explicit (IMEX) time discretizations [1].

Wang et al. [32] proposed a second order IMEX time discretization scheme for local discontinuous Galerkin method, which is unconditionally stable for the nonlinear problems, in the sense that the time-step Δt is only required to be upper-bounded by a positive constant which depends on the flow velocity and the diffusion coefficient, but is independent of the mesh size Δx. Motivated by them, we give an implementable second order time discretization for matrix-valued SDE

$$\begin{cases} dX_t^{i,j} = \left[a_1^{i,j}(X_t) + a_2^{i,j}(X_t)\right] dt + b^{i,j}(X_t)\, dW_t, \quad t > 0; \\ X_0^{i,j} = x_0^{i,j}, \end{cases} \tag{6.1}$$

where $i = 0, 1, \ldots, k$ and $j = 0, 1, \ldots, N+1$. The coefficients $a_1(\cdot)$ and $a_2(\cdot)$ come from the spatial discretization for the linear third order term u_{xxx} and the nonlinear first order term $f(u)_x$ in (1.1), respectively. In particular, for the degenerate case that $b(\cdot) \equiv 0$, our approximate scheme for SDE (6.1) given in this section coincides with the one for the ODE in [32].

We aim to use $Y_n^{i,j}$ to approximate $X_{t_n}^{i,j}$. Define $Y_0^{i,j} := x_0^{i,j}$. Suppose we already have $\{Y_n^{i,j} : i = 0, 1, \ldots, k \text{ and } j = 0, 1, \ldots, N+1\}$. Define the following operators

$$\mathcal{L}^0 f := \sum_{j=0}^{N+1} \sum_{i=0}^{k} a^{i,j} \frac{\partial f}{\partial x_{ij}} + \frac{1}{2} \sum_{l,j=0}^{N+1} \sum_{m,i=0}^{k} b^{i,j} b^{m,l} \frac{\partial^2 f}{\partial x_{ij} \partial x_{ml}},$$

and

$$\mathcal{L}^1 f := \sum_{j=0}^{N+1} \sum_{i=0}^{k} b^{i,j} \frac{\partial f}{\partial x_{ij}},$$

where and $f : \mathbb{R}^{(k+1)\times(N+2)} \longrightarrow \mathbb{R}$ is twice differentiable.

Set

$$\Delta_n = t_{n+1} - t_n, \quad \Delta W_n = W_{t_{n+1}} - W_{t_n},$$

and

$$\Delta Z_n = \int_{t_n}^{t_{n+1}} \left(W_s - W_{t_n}\right) ds, \quad \Delta U_n = \int_{t_n}^{t_{n+1}} \left(W_s - W_{t_n}\right)^2 ds.$$

6.1 Second Order Strong Taylor Scheme

As indicated in [19], we could not directly use implicit scheme for the stochastic diffusion term $b(\cdot)$. For instance, if we apply the fully implicit Euler scheme

$$Y_{n+1} = Y_n + a(Y_{n+1})\Delta_n + b(Y_{n+1})\Delta W_n, \tag{6.2}$$

to the 1-dimensional homogeneous linear SDE

$$dX_t = a\, X_t\, dt + b\, X_t\, dW_t,$$

then we obtain

$$Y_n = Y_0 \prod_{i=0}^{n-1} \frac{1}{1 - a\Delta_i - b\Delta W_i}.$$

However, this expression is not suitable as an approximation because one of its factors may become infinite. In fact, the first absolute moment $\mathbb{E}[|Y_n|]$ does not exist. It seems then that fully implicit methods involving unbounded random variables, such as (6.2), are not suitable. Thus, not only for $a_2(\cdot)$, we also consider explicit scheme for the diffusion term $b(\cdot)$, with implicit terms obtained from the corresponding Taylor approximation by suitably modifying the coefficient functions of the nonrandom multiple stochastic integrals Δ_n and Δ_n^2. Motivated by the ideas in [19, Chapter 12], we have an implicit second order strong Taylor scheme as follows

$$\begin{aligned}
Y_{n+1}^{i,j} = Y_n^{i,j} &+ a_2^{i,j}(Y_n)\Delta_n + \frac{1}{2}\mathcal{L}^0 a_2^{i,j}(Y_n)\Delta_n^2 + b^{i,j}(Y_n)\Delta W_n \\
&+ \gamma a_1^{i,j}(Y_{n+1})\Delta_n + (1-\gamma)\, a_1^{i,j}(Y_n)\Delta_n + \left(\frac{1}{2} - \gamma\right)\mathcal{L}^0 a_1^{i,j}(Y_n)\Delta_n^2 \\
&+ \frac{1}{2}\mathcal{L}^1 b^{i,j}(Y_n)\left\{(\Delta W_n)^2 - \Delta_n\right\} + \mathcal{L}^0 b^{i,j}(Y_n)\left\{\Delta W_n \Delta_n - \Delta Z_n\right\} \\
&+ \mathcal{L}^1 a_1^{i,j}(Y_n)\left\{\Delta Z_n - \gamma \Delta W_n \Delta_n\right\} + \mathcal{L}^1 a_2^{i,j}(Y_n)\Delta Z_n
\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{L}^1\mathcal{L}^1 a_2^{i,j}(Y_n)\left\{\frac{1}{2}\Delta U_n-\frac{1}{4}\Delta_n^2\right\}\\
&+\mathcal{L}^1\mathcal{L}^1 a_1^{i,j}(Y_n)\left\{\frac{1}{2}\Delta U_n-\frac{1}{4}\Delta_n^2-\frac{\gamma}{2}\Delta_n\left(\Delta W_n^2-\Delta_n\right)\right\}\\
&+\frac{1}{6}\mathcal{L}^1\mathcal{L}^1 b^{i,j}(Y_n)\left\{(\Delta W_n)^2-3\Delta_n\right\}\Delta W_n+\mathcal{L}^1\mathcal{L}^0 b^{i,j}(Y_n)\{-\Delta U_n+\Delta W_n\Delta Z_n\}\\
&+\mathcal{L}^0\mathcal{L}^1 b^{i,j}(Y_n)\left\{\frac{1}{2}\Delta U_n-\Delta W_n\Delta Z_n+\frac{1}{2}(\Delta W_n)^2\Delta_n-\frac{1}{4}\Delta_n^2\right\}\\
&+\frac{1}{24}\mathcal{L}^1\mathcal{L}^1\mathcal{L}^1 b^{i,j}(Y_n)\left\{(\Delta W_n)^4-6(\Delta W_n)^2\Delta_n+3\Delta_n^2\right\},
\end{aligned}
\tag{6.3}
$$

where $\gamma = 1-\frac{\sqrt{2}}{2}$.

6.2 Second Order Implicit–Explicit Strong Scheme

A disadvantage of the strong Taylor approximations is that the derivatives of various orders of the drift and diffusion coefficients must be evaluated at each step, in addition to the coefficients themselves. This can make implementation of such schemes a complicated undertaking. In this subsection we will propose a strong scheme which avoids the usage of derivatives in much the same way that Runge–Kutta schemes do in the deterministic setting.

6.2.1 Derivative-Free Scheme

Following the idea of [19], we could derive a second order derivative-free scheme by replacing the derivatives in the second order strong Taylor scheme (6.3) by the corresponding finite differences.

We set

$$
\begin{aligned}
\Gamma_{\pm}^{m,l} &= Y_n^{m,l}+a^{m,l}(Y_n)\Delta_n \pm b^{m,l}(Y_n)\sqrt{\Delta_n},\\
\eta_{\pm}^{m,l} &= Y_n^{m,l}\pm b^{m,l}(Y_n)\Delta_n;\\
\phi_{+,\pm}^{m,l} &= \Gamma_{+}^{m,l}+a^{m,l}(\Gamma_+)\Delta_n \pm b^{m,l}(\Gamma_+)\sqrt{\Delta_n},\\
\phi_{-,\pm}^{m,l} &= \Gamma_{-}^{m,l}+a^{m,l}(\Gamma_-)\Delta_n \pm b^{m,l}(\Gamma_-)\sqrt{\Delta_n};\\
\beta_{+,\pm}^{m,l} &= \phi_{+,+}^{m,l}\pm b^{m,l}(\phi_{+,+})\sqrt{\Delta_n},\\
\beta_{-,\pm}^{m,l} &= \phi_{+,-}^{m,l}\pm b^{m,l}(\phi_{+,-})\sqrt{\Delta_n};\\
\theta_{\pm}^{m,l} &= Y_n^{m,l}+\gamma a_1^{m,l}(\theta_\pm)\Delta_n+\gamma a_2^{m,l}(Y_n)\Delta_n \pm b^{m,l}(Y_n)\sqrt{\gamma\Delta_n}.
\end{aligned}
\tag{6.4}
$$

For a sufficiently smooth function $f:\mathbb{R}^{(k+1)\times(N+2)}\longrightarrow\mathbb{R}$, we have

$$
\begin{aligned}
\mathcal{L}^1 f^{i,j}(Y_n) &= \frac{1}{2\Delta_n}\left\{f^{i,j}(\eta_+)-f^{i,j}(\eta_-)\right\}+\mathcal{O}(\Delta_n^2),\\
\mathcal{L}^1 f^{i,j}(Y_n) &= \frac{1}{2\sqrt{\Delta_n}}\left\{f^{i,j}(\Gamma_+)-f^{i,j}(\Gamma_-)\right\}+\mathcal{O}(\Delta_n),\\
\mathcal{L}^0 f^{i,j}(Y_n) &= \frac{1}{2\Delta_n}\left\{f^{i,j}(\Gamma_+)-2f^{i,j}(Y_n)+f^{i,j}(\Gamma_-)\right\}+\mathcal{O}(\Delta_n),\\
f^{i,j}(Y_n)+\gamma\mathcal{L}^0 f^{i,j}(Y_n)\Delta_n &= \frac{1}{2}\left[f^{i,j}(\theta_+)+f^{i,j}(\theta_-)\right]+\mathcal{O}(\Delta_n^{\frac{3}{2}}),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^1\mathcal{L}^1 f^{i,j}(Y_n) &= \frac{1}{4\Delta_n}\left\{f^{i,j}(\phi_{+,+}) - f^{i,j}(\phi_{+,-}) - f^{i,j}(\phi_{-,+})\right.\\
&\quad \left.+ f^{i,j}(\phi_{-,-})\right\} + \mathcal{O}(\Delta_n),\\
\mathcal{L}^1\mathcal{L}^1 f^{i,j}(Y_n) &= \frac{1}{2\Delta_n}\left\{f^{i,j}(\phi_{+,+}) - f^{i,j}(\phi_{+,-}) - f^{i,j}(\Gamma_{+})\right.\\
&\quad \left.+ f^{i,j}(\Gamma_{-})\right\} + \mathcal{O}(\sqrt{\Delta_n}),\\
\mathcal{L}^1\mathcal{L}^0 f^{i,j}(Y_n) &= \frac{1}{2\Delta_n^{\frac{3}{2}}}\left\{f^{i,j}(\phi_{+,+}) + f^{i,j}(\phi_{+,-}) - 3f^{i,j}(\Gamma_{+})\right.\\
&\quad \left.- f^{i,j}(\Gamma_{-}) + 2f^{i,j}(Y_n)\right\} + \mathcal{O}(\sqrt{\Delta_n}),\\
\mathcal{L}^0\mathcal{L}^1 f^{i,j}(Y_n) &= \frac{1}{4\Delta_n^{\frac{3}{2}}}\left\{f^{i,j}(\phi_{+,+}) - f^{i,j}(\phi_{+,-}) + f^{i,j}(\phi_{-,+})\right.\\
&\quad \left.- f^{i,j}(\phi_{-,-}) - 2f^{i,j}(\Gamma_{+}) + 2f^{i,j}(\Gamma_{-})\right\} + \mathcal{O}(\sqrt{\Delta_n}),\\
\mathcal{L}^1\mathcal{L}^1\mathcal{L}^1 f^{i,j}(Y_n) &= \frac{1}{4\Delta_n^{\frac{3}{2}}}\left\{f^{i,j}(\beta_{+,+}) - f^{i,j}(\beta_{+,-}) - f^{i,j}(\beta_{-,+})\right.\\
&\quad + f^{i,j}(\beta_{-,-}) - f^{i,j}(\phi_{+,+})\\
&\quad \left.+ f^{i,j}(\phi_{+,-}) + f^{i,j}(\phi_{-,+}) - f^{i,j}(\phi_{-,-})\right\} + \mathcal{O}(\sqrt{\Delta_n}).
\end{aligned}
$$

Then scheme (6.3) reads

$$
\begin{aligned}
Y_{n+1}^{i,j} &= Y_n^{i,j} + \delta a_2^{i,j}(Y_n)\Delta_n + \frac{1}{2}(1-\delta)\left\{a_2^{i,j}(\theta_+) + a_2^{i,j}(\theta_-)\right\}\Delta_n + b^{i,j}(Y_n)\Delta W_n\\
&\quad + \gamma a_1^{i,j}(Y_{n+1})\Delta_n + \frac{1}{2}(1-\gamma)\left\{a_1^{i,j}(\theta_+) + a_1^{i,j}(\theta_-)\right\}\Delta_n\\
&\quad + \frac{1}{4\Delta_n}\left\{b^{i,j}(\eta_+) - b^{i,j}(\eta_-)\right\}\left\{(\Delta W_n)^2 - \Delta_n\right\}\\
&\quad + \frac{1}{2\Delta_n}\left\{b^{i,j}(\Gamma_+) - 2b^{i,j}(Y_n) + b^{i,j}(\Gamma_-)\right\}\left\{\Delta W_n\Delta_n - \Delta Z_n\right\}\\
&\quad + \frac{1}{2\sqrt{\Delta_n}}\left\{a_1^{i,j}(\Gamma_+) - a_1^{i,j}(\Gamma_-)\right\}\left\{\Delta Z_n - \gamma\Delta W_n\Delta_n\right\}\\
&\quad + \frac{1}{2\sqrt{\Delta_n}}\left\{a_2^{i,j}(\Gamma_+) - a_2^{i,j}(\Gamma_-)\right\}\Delta Z_n\\
&\quad + \frac{1}{2\Delta_n}\left\{a_2^{i,j}(\phi_{+,+}) - a_2^{i,j}(\phi_{+,-}) - a_2^{i,j}(\Gamma_+) + a_2^{i,j}(\Gamma_-)\right\}\left\{\frac{1}{2}\Delta U_n - \frac{1}{4}\Delta_n^2\right\}\\
&\quad + \frac{1}{2\Delta_n}\left\{a_1^{i,j}(\phi_{+,+}) - a_1^{i,j}(\phi_{+,-}) - a_1^{i,j}(\Gamma_+) + a_1^{i,j}(\Gamma_-)\right\}\\
&\quad \times\left\{\frac{1}{2}\Delta U_n - \frac{1}{4}\Delta_n^2 - \frac{\gamma}{2}\Delta_n\left(\Delta W_n^2 - \Delta_n\right)\right\}\\
&\quad + \frac{1}{8\Delta_n}\left\{b^{i,j}(\phi_{+,+}) - b^{i,j}(\phi_{+,-}) - b^{i,j}(\phi_{-,+}) + b^{i,j}(\phi_{-,-})\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left\{ \frac{1}{3} (\Delta W_n)^2 - \Delta_n \right\} \Delta W_n \\
&+ \frac{1}{2\Delta_n^{\frac{3}{2}}} \left\{ b^{i,j}(\phi_{+,+}) + b^{i,j}(\phi_{+,-}) - 3b^{i,j}(\Gamma_+) - b^{i,j}(\Gamma_-) + 2b^{i,j}(Y_n) \right\} \\
&\times \{ -\Delta U_n + \Delta W_n \Delta Z_n \} \\
&+ \frac{1}{4\Delta_n^{\frac{3}{2}}} \Big\{ b^{i,j}(\phi_{+,+}) - b^{i,j}(\phi_{+,-}) + b^{i,j}(\phi_{-,+}) - b^{i,j}(\phi_{-,-}) \\
&- 2b^{i,j}(\Gamma_+) + 2b^{i,j}(\Gamma_-) \Big\} \\
&\times \left\{ \frac{1}{2} \Delta U_n - \Delta W_n \Delta Z_n + \frac{1}{2} (\Delta W_n)^2 \Delta_n - \frac{1}{4} \Delta_n^2 \right\} \\
&+ \frac{1}{96\Delta_n^{\frac{3}{2}}} \Big\{ b^{i,j}(\beta_{+,+}) - b^{i,j}(\beta_{+,-}) - b^{i,j}(\beta_{-,+}) + b^{i,j}(\beta_{-,-}) \\
&- b^{i,j}(\phi_{+,+}) + b^{i,j}(\phi_{+,-}) + b^{i,j}(\phi_{-,+}) - b^{i,j}(\phi_{-,-}) \Big\} \\
&\times \left\{ (\Delta W_n)^4 - 6 (\Delta W_n)^2 \Delta_n + 3\Delta_n^2 \right\},
\end{aligned}
\tag{6.5}
$$

where $\delta = 1 - \frac{1}{2\gamma}$.

6.2.2 Modeling of the Itô Integrals

We have proposed a derivative-free scheme (6.5). Now it remains to model at each step three random variables ΔW_n, ΔZ_n and ΔU_n. In [25], the characteristic function of these random variables is found. However, it is very complicated and cannot be easily used in practice. Thus, the exact modeling has poor perspectives, and therefore we need to be able to model these variables approximately. The detailed method of modeling can be found in [26].

Introduce the new process

$$
v(s) = \frac{W_{t_n + \Delta_n s} - W_{t_n}}{\sqrt{\Delta_n}}, \quad 0 \le s \le 1.
$$

It is obvious that $\{v(s),\ 0 \le s \le 1\}$ is a standard Wiener process. We have

$$
\Delta W_n = \Delta_n^{\frac{1}{2}} v(1), \quad \Delta Z_n = \Delta_n^{\frac{3}{2}} \int_0^1 v(s)\, ds, \quad \Delta U_n = \Delta_n^2 \int_0^1 v^2(s)\, ds.
$$

Then the problem of modeling the random variables ΔW_n, ΔZ_n and ΔU_n could be reduced to that of modeling the variables $v(1)$, $\int_0^1 v(s)\, ds$ and $\int_0^1 v^2(s)\, ds$. These variables are the solution of the system of equations

$$
\begin{cases}
dx = dv(s), & x(0) = 0, \\
dy = x\, ds, & y(0) = 0, \\
dz = x^2\, ds, & z(0) = 0,
\end{cases}
\tag{6.6}
$$

at the moment $s = 1$.

Let $x_k = \bar{x}(s_k)$, $y_k = \bar{y}(s_k)$, $z_k = \bar{z}(s_k)$, $0 = s_0 < s_1 < \cdots < s_{N_n} = 1$, $s_{k+1} - s_k = \delta_n = \frac{1}{N_n}$, be an approximate solution of (6.6), where N_n is to be determined. We will now use a method of order 1.5 to integrate (6.6).

$$\begin{cases} x_{k+1} = x_k + (v(s_{k+1}) - v(s_k)), \\ y_{k+1} = y_k + x_k\delta_n + \displaystyle\int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k))\, d\theta, \\ z_{k+1} = z_k + x_k^2\delta_n + 2x_k \displaystyle\int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k))\, d\theta + \frac{\delta_n^2}{2}. \end{cases} \tag{6.7}$$

The pair of correlated normally distributed random variables $v(s_{k+1}) - v(s_k)$ and $\int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k))\, d\theta$ are generated by

$$v(s_{k+1}) - v(s_k) = \zeta_{k,1}\delta_n^{\frac{1}{2}}, \qquad \int_{s_k}^{s_{k+1}} (v(\theta) - v(s_k))\, d\theta = \frac{1}{2}\left(\zeta_{k,1} + \frac{1}{\sqrt{3}}\zeta_{k,2}\right)\delta_n^{\frac{3}{2}}, \tag{6.8}$$

where $\zeta_{k,1}$ and $\zeta_{k,2}$ are independent normally $N(0; 1)$ distributed random variables.

We choose δ_n such that $\delta_n = \mathcal{O}(\Delta_n^{\frac{1}{3}})$ i.e.

$$N_n = \left\lceil \Delta_n^{-\frac{1}{3}} \right\rceil, \tag{6.9}$$

with $\lceil\cdot\rceil$ standing for the ceiling function.

Then we have $\Delta_n^{\frac{1}{2}} x_{N_n} = \Delta W_n$, $\Delta_n^{\frac{3}{2}} y_{N_n} = \Delta Z_n$ and

$$\left(\mathbb{E}\left[\left|\Delta_n^2 z_{N_n} - \Delta U_n\right|^2\right]\right)^{\frac{1}{2}} = \mathcal{O}\left(\Delta_n^{\frac{5}{2}}\right).$$

Thus according to [26, Theorem 4.2, page 50], in a method of second order of accuracy with time step Δ_n such as scheme (6.5), we could replace ΔW_n, ΔZ_n and ΔU_n by $\Delta_n^{\frac{1}{2}} x_{N_n}$, $\Delta_n^{\frac{3}{2}} y_{N_n}$ and $\Delta_n^2 z_{N_n}$ independently at each step. Finally, we get an implementable second order derivative-free time discretization scheme,

$$\begin{aligned} Y_{n+1}^{i,j} &= Y_n^{i,j} + \delta a_2^{i,j}(Y_n)\Delta_n + \frac{1}{2}(1-\delta)\left\{a_2^{i,j}(\theta_+) + a_2^{i,j}(\theta_-)\right\}\Delta_n + b^{i,j}(Y_n)x_{N_n}\sqrt{\Delta_n} \\ &+\gamma a_1^{i,j}(Y_{n+1})\Delta_n + \frac{1}{2}(1-\gamma)\left\{a_1^{i,j}(\theta_+) + a_1^{i,j}(\theta_-)\right\}\Delta_n \\ &+\frac{1}{4}\left\{b^{i,j}(\eta_+) - b^{i,j}(\eta_-)\right\}\left\{x_{N_n}^2 - 1\right\} \\ &+\frac{1}{2}\left\{b^{i,j}(\Gamma_+) - 2b^{i,j}(Y_n) + b^{i,j}(\Gamma_-)\right\}\left\{x_{N_n} - y_{N_n}\right\}\sqrt{\Delta_n} \\ &+\frac{1}{2}\left\{a_1^{i,j}(\Gamma_+) - a_1^{i,j}(\Gamma_-)\right\}\left\{y_{N_n} - \gamma x_{N_n}\right\}\Delta_n \\ &+\frac{1}{2}\left\{a_2^{i,j}(\Gamma_+) - a_2^{i,j}(\Gamma_-)\right\}y_{N_n}\Delta_n \\ &+\frac{1}{4}\left\{a_2^{i,j}(\phi_{+,+}) - a_2^{i,j}(\phi_{+,-}) - a_2^{i,j}(\Gamma_+) + a_2^{i,j}(\Gamma_-)\right\}\left\{z_{N_n} - \frac{1}{2}\right\}\Delta_n \\ &+\frac{1}{4}\left\{a_1^{i,j}(\phi_{+,+}) - a_1^{i,j}(\phi_{+,-}) - a_1^{i,j}(\Gamma_+) + a_1^{i,j}(\Gamma_-)\right\} \\ &\times\left\{z_{N_n} - \frac{1}{2} + \gamma - \gamma x_{N_n}^2\right\}\Delta_n \\ &+\frac{1}{8}\left\{b^{i,j}(\phi_{+,+}) - b^{i,j}(\phi_{+,-}) - b^{i,j}(\phi_{-,+}) + b^{i,j}(\phi_{-,-})\right\} \end{aligned}$$

$$
\begin{aligned}
&\times \left\{\frac{1}{3}x_{N_n}^2 - 1\right\} x_{N_n}\sqrt{\Delta_n} \\
&+ \frac{1}{2}\left\{b^{i,j}(\phi_{+,+}) + b^{i,j}(\phi_{+,-}) - 3b^{i,j}(\Gamma_+) - b^{i,j}(\Gamma_-) + 2b^{i,j}(Y_n)\right\} \\
&\times \left\{x_{N_n}y_{N_n} - z_{N_n}\right\}\sqrt{\Delta_n} \\
&+ \frac{1}{4}\left\{b^{i,j}(\phi_{+,+}) - b^{i,j}(\phi_{+,-}) + b^{i,j}(\phi_{-,+}) - b^{i,j}(\phi_{-,-})\right. \\
&\left. - 2b^{i,j}(\Gamma_+) + 2b^{i,j}(\Gamma_-)\right\} \\
&\times \left\{\frac{1}{2}z_{N_n} - x_{N_n}y_{N_n} + \frac{1}{2}x_{N_n}^2 - \frac{1}{4}\right\}\sqrt{\Delta_n} \\
&+ \frac{1}{96}\left\{b^{i,j}(\beta_{+,+}) - b^{i,j}(\beta_{+,-}) - b^{i,j}(\beta_{-,+}) + b^{i,j}(\beta_{-,-})\right. \\
&\left. - b^{i,j}(\phi_{+,+}) + b^{i,j}(\phi_{+,-}) + b^{i,j}(\phi_{-,+}) - b^{i,j}(\phi_{-,-})\right\} \\
&\times \left\{x_{N_n}^4 - 6x_{N_n}^2 + 3\right\}\sqrt{\Delta_n},
\end{aligned} \tag{6.10}
$$

where $x_{N_n}, y_{N_n}, z_{N_n}$ are computed by (6.7)–(6.9), and $\Gamma_\pm, \eta_\pm, \theta_\pm, \phi_{\pm,\pm}, \beta_{\pm,\pm}$ are calculated by (6.4).

6.3 Numerical Tests for IMEX Time Discretization

Now we apply the time discretization (6.10) to some SDEs for verifying the second-order accuracy of the IMEX scheme. The positive real number T is the terminal time and the time-step is given by $\Delta t = T/N_T$. We use $M = 15{,}000$ realizations for Monte Carlo technique to approximate the $L^2(\Omega)$-errors

$$
\mathbb{E}\left[\left|Y_{N_T} - X_T\right|^2\right] \approx e_2^2 \pm \mathcal{V},
$$

with

$$
e_2 := \left(\frac{1}{M}\sum_{i=1}^{M} z_i\right)^{\frac{1}{2}}, \qquad \mathcal{V} := \frac{2}{\sqrt{M}}\left[\frac{1}{M}\sum_{i=1}^{M} z_i^2 - \left(\frac{1}{M}\sum_{i=1}^{M} z_i\right)^2\right]^{\frac{1}{2}},
$$

where $z_i := \left|Y_{N_T}(\omega_i) - X_T(\omega_i)\right|^2$, $Y_{N_T}(\omega_i)$ is one simulation from M paths, and $X_T(\omega_i)$ is the exact solution with the corresponding path ω_i.

6.3.1 Linear Case

Let us first consider the following linear SDEs:

$$
\begin{cases} dX_t = (c_1X_t + c_2X_t)\,dt + c_3X_t\,dW_t & (\omega, t) \in \Omega \times (0, T], \\ X_0 = x_0, & \omega \in \Omega, \end{cases} \tag{6.11}
$$

where c_1, c_2, c_3, x_0 are fixed real numbers. The exact solution of (6.11) is

$$
X_t(\omega) = x_0\, e^{c_1t + c_2t + c_3W_t(\omega) - \frac{1}{2}c_3^2t}.
$$

Table 1 Accuracy on (6.11) with $M = 15{,}000$, $c_1 = -1.5$, $c_2 = -1.0$, $c_3 = x_0 = 1.0$, $T = 0.1$

N_T	e_2	Order	$\mathcal{V}$
10	8.58E−05	–	1.94E−10
20	2.13E−05	2.01	1.31E−11
40	5.32E−06	2.00	9.44E−13
80	1.36E−06	1.96	6.76E−14
160	3.42E−07	2.00	4.21E−15
320	8.48E−08	2.01	2.80E−16

Table 2 Accuracy on (6.12) with $M = 15{,}000$, $c_4 = 1.0$, $c_5 = -1.0$, $T = 0.1$

N_T	e_2	Order	$\mathcal{V}$
10	2.13E−05	–	2.50E−11
20	5.42E−06	1.98	1.30E−12
40	1.38E−06	1.98	8.26E−14
80	3.39E−07	2.02	4.16E−15
160	8.60E−08	1.98	4.61E−16
320	2.11E−08	2.03	1.68E−17

In this case, we have

$$a_1(x) = c_1 x, \qquad a_2(x) = c_2 x, \qquad b(x) = c_3 x.$$

We use IMEX scheme (6.10) on Eq. (6.11), in which we use implicit scheme for $a_1(\cdot)$ and explicit scheme for $a_2(\cdot)$ and $b(\cdot)$. In Table 1, we show the errors and order of accuracy with $c_1 = -1.5$, $c_2 = -1.0$, $c_3 = 1.0$, $x_0 = 1.0$ and $T = 0.1$. We could observe that the scheme has second-order accuracy.

6.3.2 Nonlinear Case

Next we test the IMEX scheme (6.10) on the following nonlinear SDEs:

$$\begin{cases} dX_t = \left(-\frac{1}{2}c_4^2\, X_t + c_5\sqrt{1 - X_t^2}\right) dt + c_4\sqrt{1 - X_t^2}\, dW_t, & (\omega, t) \in \Omega \times (0, T], \\ X_0 = 0, & \omega \in \Omega, \end{cases} \tag{6.12}$$

where c_4, c_5 are fixed real numbers. The exact solution of (6.12) is

$$X_t(\omega) = \sin\left(c_4\, W_t(\omega) + c_5\, t\right).$$

In this case, we have

$$a_1(x) = -\frac{1}{2}c_4^2\, x, \qquad a_2(x) = c_5\sqrt{1 - x^2}, \qquad b(x) = c_4\sqrt{1 - x^2}.$$

We apply IMEX scheme (6.10) to Eq. (6.12), in which we use implicit scheme for linear term $a_1(\cdot)$ and explicit scheme for nonlinear terms $a_2(\cdot)$ and $b(\cdot)$. In Table 2, we show the errors and order of accuracy with $c_4 = 1.0$, $c_5 = -1.0$ and $T = 0.1$. We could see that the scheme has second-order accuracy.

7 Numerical Experiments

In this section we consider the application of the numerical method, which we have defined in Sect. 3, on some model problems. Here, M is the number of realizations. The positive real number T is the terminal time. In Theorem 5.1, the error estimate is given by using the $L^2(\Omega \times [0, 2\pi] \times [0, T])$-norm. Since the mathematical expectation could not be calculated exactly, the $L^2(\Omega \times [0, 2\pi] \times [0, T])$-errors are approximated by the Monte Carlo technique

$$\mathbb{E}\left[\|u_h(\cdot, \cdot, T) - u(\cdot, \cdot, T)\|^2_{L^2(0,2\pi)}\right] \approx e_2^2 \pm \mathcal{V} \ ,$$

with

$$e_2 := \left(\frac{1}{M}\sum_{i=1}^{M} z_i\right)^{\frac{1}{2}}, \qquad \mathcal{V} := \frac{2}{\sqrt{M}}\left[\frac{1}{M}\sum_{i=1}^{M} z_i^2 - \left(\frac{1}{M}\sum_{i=1}^{M} z_i\right)^2\right]^{\frac{1}{2}},$$

where $z_i := \|u_h(\omega_i, \cdot, T) - u(\omega_i, \cdot, T)\|^2_{L^2(0,2\pi)}$, $u_h(\omega_i, \cdot, T)$ is one simulation from M paths, and $u(\omega_i, \cdot, T)$ is the exact solution with the corresponding path ω_i. We use e_2 to approximate the L^2 error. The quantity $\mathcal{V}$ is called the statistical error. The run-time T_R (in seconds) showed in all tables is the CPU running time for computation of M realizations (with 16 cores for parallel computing). The degree of the piecewise-polynomial space V_h is k. Since we use the implicit time-marching in this paper, the stringent stability condition $\Delta t \sim (\Delta x)^3$ can be removed, which is necessary for third-order PDEs if one uses explicit time discretization. In all experiments of ultra-weak DG scheme, we adjust the time step to $\Delta t \sim (\Delta x)^{\frac{k+1}{2}}$ so that the time discretization is effectively $(k+1)$-th order of accuracy.

7.1 Linear Stochastic Third-Order Equation

We consider the following linear third-order equation

$$\begin{cases} du = -u_{xxx}\, dt + bu\, dW_t & \text{in } \Omega \times [0, 2\pi] \times (0, T), \\ u(\omega, x, 0) = \sin(x), & \omega \in \Omega, x \in [0, 2\pi]. \end{cases} \tag{7.1}$$

The exact solution of (7.1) is

$$u(\omega, x, t) = \sin(x + t)e^{bW_t(\omega) - \frac{1}{2}b^2 t}.$$

In Table 3, we show L^2-errors for the linear Eq. (7.1). Our computation is based on the flux choice (3.2) and (3.3). We observe that our scheme is not consistent for P^1 polynomials, while optimal $(k+1)$-th order of accuracy is achieved for $k \geq 2$. The results on the run-time show clearly that the ultra-weak DG scheme with $k = 3$ is more efficient than the one with $k = 2$ to reach the same error levels. All the numerical results coincide with the conclusion of Theorem 5.1.

7.2 Linear Stochastic KdV Equations

In the following we test the accuracy of the ultra-weak DG method on the linear stochastic KdV equations as follows,

$$\begin{cases} du = -\left(u_{xxx} - u_x\right) dt + b\, u\, dW_t & \text{in } \Omega \times [0, 2\pi] \times (0, T), \\ u(\omega, x, 0) = \sin(x), & \omega \in \Omega, x \in [0, 2\pi]. \end{cases} \tag{7.2}$$

Table 3 Accuracy on (7.1) with $b = 1.0$, $T = 0.01$, $M = 1000$

N	e_2	Order	$\mathcal{V}$	T_R
$k = 1$				
10	9.37E−02	–	1.11E−04	0.56
20	1.67E−01	−0.84	3.64E−04	0.59
40	9.41E−02	0.83	1.13E−04	0.79
80	3.12E−02	1.59	1.22E−05	1.97
160	2.76E−02	0.18	9.49E−06	16.75
$k = 2$				
10	1.45E−02	–	2.70E−06	0.67
20	2.65E−03	2.45	8.77E−08	0.92
40	3.27E−04	3.02	1.43E−09	1.48
80	4.08E−05	3.00	2.05E−11	11.39
160	5.11E−06	3.00	3.35E−13	343.55
$k = 3$				
10	5.59E−04	–	3.90E−09	0.67
20	3.62E−05	3.95	1.66E−11	1.13
40	2.27E−06	3.99	6.61E−14	3.24
80	1.42E−07	4.00	2.58E−16	69.13
160	8.90E−09	4.00	1.02E−18	2799.75

The exact solution of (7.2) is

$$u(\omega, x, t) = sin\,(x + 2t)\, e^{bW_t(\omega) - \frac{1}{2}b^2 t}.$$

We still use (3.2) and (3.3) as our flux choice and take the upwind flux for the first order convection term $f(u) = -u$, i.e. $\widehat{f}(u^-, u^+) = -u^+$. The errors and numerical order of accuracy for P^k elements with $1 \le k \le 3$ are listed in Table 4, which show that our scheme gives the optimal $(k+1)$-th order of accuracy when $k \ge 2$. For P^1, the scheme is not consistent. The scheme with $k = 3$ is more efficient than the one with $k = 2$.

7.3 Stochastic Nonlinear KdV Equations

Although we could not give error estimates for fully nonlinear equations, it is worth trying to apply the ultra-weak DG method to solve some nonlinear stochastic equations. The next example is the stochastic nonlinear KdV equations,

$$\begin{cases} du = -\left[u_{xxx} + 3\dfrac{\partial}{\partial x}\left(u^2\right)\right] dt + b\, dW_t & \text{in } \Omega \times [0, 2\pi] \times (0, T), \\ u(\omega, x, 0) = \sin(x), & \omega \in \Omega, x \in [0, 2\pi]. \end{cases} \tag{7.3}$$

The exact solution of (7.3) is

$$u(\omega, x, t) = v\left(x - 6\, b \int_0^t W_s\, ds,\ t\right) + bW_t, \tag{7.4}$$

Table 4 Accuracy on (7.2) with $b = 1.0$, $T = 0.01$, $M = 1000$

N	e_2	Order	$\mathcal{V}$	T_R
$k = 1$				
10	8.74E−02	–	9.66E−05	0.70
20	1.52E−01	−0.80	3.02E−04	0.73
40	8.80E−02	0.79	9.87E−05	0.82
80	3.71E−02	1.25	1.72E−05	2.38
160	2.70E−02	0.46	9.08E−06	22.30
$k = 2$				
10	1.43E−02	–	2.63E−06	0.71
20	2.63E−03	2.44	8.67E−08	1.03
40	3.26E−04	3.01	1.43E−09	1.83
80	4.08E−05	3.00	2.05E−11	15.98
160	5.11E−06	3.00	3.35E−13	444.89
$k = 3$				
10	5.68E−04	–	4.15E−09	0.71
20	3.63E−05	3.97	1.69E−11	1.32
40	2.27E−06	4.00	6.37E−14	4.55
80	1.43E−07	3.99	2.60E−16	107.46
160	8.88E−09	4.01	1.00E−18	3689.89

where v is the solution of the following deterministic nonlinear KdV equations

$$\begin{cases} v_t + v_{xxx} + 3\dfrac{\partial}{\partial x}\left(v^2\right) = 0 & \text{in } \Omega \times [0, 2\pi] \times (0, T), \\ v(\omega, x, 0) = \sin(x), & \omega \in \Omega, x \in [0, 2\pi]. \end{cases} \tag{7.5}$$

We use (3.2) and (3.3) as our flux. For the first order nonlinear convection term $f(u) = 3u^2$, we use the simple Lax-Friedrichs flux

$$\widehat{f}\left(u^-, u^+\right) = \frac{3}{2}\left\{\left(u^-\right)^2 + \left(u^+\right)^2\right\} - 3\alpha\left(u^+ - u^-\right),$$

where

$$\alpha = \max_j \left\{\left|u^-_{j+\frac{1}{2}}\right|, \left|u^+_{j+\frac{1}{2}}\right|\right\}.$$

In Table 5, we show the L^2-errors and order of accuracy for Eq. (7.3). We could see that the order of accuracy converges to $k + 1$ when $k \geq 2$. The scheme lose the order of accuracy when $k = 1$. The scheme with $k = 3$ is more efficient than the one with $k = 2$.

Remark 7.1 For the SPDEs driven by an additive noise, unlike the diffusion effect of the stochastic terms on the solutions to (7.1) and (7.2), here the stochastic term only has the shift effect on the solution of (7.3) since the stochastic perturbation in (7.4) is additive. Thus the value of b has little influence on the error and $M = 100$ is good enough to approximate the mathematical expectation. On the other hand, the cost for the computation of nonlinear equations is quite high, so it would cost too much to compute the nonlinear case with $M = 1000$.

Table 5 Accuracy on (7.3) with $b = 1.0$, $T = 0.1$, $M = 100$

N	e_2	Order	$\mathcal{V}$	T_R
$k = 1$				
10	3.22E−01	–	9.86E−04	0.26
20	3.37E−01	−0.06	7.65E−04	0.34
40	3.62E−01	−0.10	1.23E−03	1.15
80	3.75E−01	−0.05	7.39E−04	2.64
160	3.77E−01	−0.01	5.24E−04	11.33
$k = 2$				
10	9.42E−02	–	3.07E−04	1.41
20	2.68E−02	1.82	1.20E−05	2.98
40	4.61E−03	2.54	1.13E−07	14.42
80	6.18E−04	2.90	5.44E−10	81.67
160	7.84E−05	2.98	2.38E−12	611.12
$k = 3$				
10	8.75E−03	–	2.75E−06	2.50
20	5.37E−04	4.03	4.70E−10	12.52
40	3.31E−05	4.02	5.65E−13	89.32
80	2.05E−06	4.01	5.40E−16	747.71
160	1.28E−07	4.00	1.75E−18	7581.41

8 Concluding Remarks

In this article, we present an ultra-weak DG scheme for generalized stochastic KdV equations. The $L^2(0, 2\pi)$-stability result of the scheme is obtained, and the optimal error estimate of order $\mathcal{O}(h^{k+1})$ with respect to spatial L^2-norm for semilinear stochastic equations is proved. We combine a second order implicit–explicit derivative-free time discretization scheme, which could reduce the computational costs, to perform several numerical experiments on some model problems to confirm the analytical results. Even though we concentrate on the one-dimensional case in this paper, the numerical algorithm and its stability analysis can be generalized to higher dimensions straightforwardly. But the optimal error estimates for multi-dimensional case will be more involved, especially on unstructured meshes. In the future, we would like to investigate error estimates for fully nonlinear stochastic equations in higher spatial dimensional settings with unstructured meshes.

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