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BMO martingale method for backward stochastic differential equations driven by general càdlàg local martingales

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In this paper we study time-discontinuous nonlinear multi-dimensional backward stochastic differential equations (BSDEs) driven by general càdlàg local martingales. The Lipschitz coefficients of the generators are allowed to be unbounded. The time-discontinuous BMO martingale theory, in particular Fefferman's inequality, is used to study the existence and uniqueness of solution in $\mathcal{S}^p$ with $p \in (1, \infty]$.

Keywords and phrases: Backward stochastic differential equations, càdlàg local martingale, time-discontinuous BMO martingale theory, Fefferman's inequality.

1. Introduction

Backward stochastic differential equations (BSDEs) are widely connected to various fields, such as stochastic control and optimization, mathematical finance, theoretical economics, partial differential equations, differential geometry. See among others [8, 12] and the references therein. In this paper, we use the time-discontinuous BMO martingale theory to study the following multi-dimensional nonlinear BSDEs with jumps:

$$
\begin{aligned}
Y_t = \xi + J_T - J_t + \int_{t+}^T f(s, Y_{s-}) d[N_1, N_2]_s + \int_{t+}^T g(s, Y_{s-}, Z_s) d[N_3, M]_s \\
- \int_{t+}^T Z_s \, dM_s - \int_{t+}^T dM_s^{\perp}, \qquad t \in [0, T),
\end{aligned} \tag{1}
$$

where J is a càdlàg process, N_1, N_2, N_3, and M are general càdlàg local martingales, and $M^{\perp}$ is strongly orthogonal to M.

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BSDEs are introduced by Bismut [2, 3, 4], in particular in its linear form as an adjoint equation in the Pontryagin stochastic maximum principle and in a nonlinear form as the backward stochastic Riccati equation (an equivalent matrix form of the stochastic Bellman equation) for the stochastic linear quadratic optimal control problem. Pardoux and Peng [18] proved the seminal existence and uniqueness theorem for nonlinear Lipschitz continuous BSDEs. Since then, BSDEs have been studied in different spaces of solutions under different assumptions on the generators (see e.g. [5, 9] and the references therein).

There are numerous efforts at the solution of BSDEs driven by discontinuous local martingales. Tang and Li [21] obtain the existence of a unique $\mathcal{S}^2$ solution to a BSDE driven by a Poisson random measure independent of the Brownian motion. For $p \in (1, 2)$, Yao [22] shows that the above BSDE admits a unique $\mathcal{S}^p$ solution by approximating the monotonic generator by a sequence of Lipschitz generators via convolution. By introducing stronger integrability condition on the terminal value, Buckdahn [6] and El Karoui and Huang [11] consider a general BSDE driven by a general càdlàg martingale and continuous increasing process in the generalized sense. Carbone et al. [7] consider the above BSDE in the space $\mathcal{S}^2$, where the solution has the same power $p = 2$ of integrability as the terminal value, but require that the generators are uniformly Lipschitz continuous. Recently, Papapantoleon et al. [17] propose a wellposedness result for BSDE with possibly unbounded random time horizon and driven by a general martingale in a filtration that may be stochastically discontinuous. See also [1, 13, 10, 16, 20] and the references therein for BSDEs with jumps. However, solution of BSDEs driven by general càdlàg local martingales, with the same power p of integrability as the underlying data (ξ, J) for $p \in (1, \infty]$, still remains to be studied.

In 2010, Delbaen and Tang [9] use the theory of BMO martingales to prove the unique solvability of BSDEs (1) for continuous local martingale M, where the adapted solutions have the same power p of integrability to the underlying data for $p \in (1, \infty]$. In this paper, we extend the preceding work to allow M to be discontinuous. We have to develop some inequalities with the discontinuous BMO martingale theory, which are essential to deal with the unbounded Lipschitz coefficients and càdlàg driving terms. In contrast to the case of continuous M, BSDEs (1) driven by càdlàg local martingales have at least the following three novelties arising from the appearance of the jump: Firstly, since the quadratic variation of a discontinuous process with bounded total variation is no longer vanishing, the resulting new terms have to be well dominated; Secondly, many useful tools in Kazamaki [15] have to be adapted to our more general BMO martingale M (see the next section for details); Thirdly, in the discontinuous case, the covariation process between two orthogonal processes M and $M^{\perp}$ is only a local martingale, and is not necessarily equal to *zero* any more (while $[M, M^{\perp}] = \langle M, M^{\perp}\rangle = 0$ holds in the continuous case), which leads us to consider BSDEs of the adjusted form (9) for $p \neq 2$ and to introduce the condition of extreme point of $\Gamma(M)$ to guarantee the strong property of predictable representation.

The rest of the paper is organized as follows. Section 2 consists of three subsections. In the first two subsections, we provide some basic notations, definitions and well-known inequalities. Fefferman's inequality is crucial in this paper. It will be used to prove some inequalities and properties in Subsection 2.3, which are essential to the proof of our main results. Section 3 consists of three subsections. In Subsection 3.1, we propose the unique existence result in $\mathcal{S}^2 \times (\mathcal{H}^2)^2$ under some suitable sliceability assumption. In Subsection 3.2, we obtain a new existence result in $\mathcal{S}^\infty \times (BMO)^2$ when the data $(J, \xi) \in \mathcal{S}^\infty \times L^\infty$. In Subsection 3.3, we study the general case of $p \in (1, \infty)$. In Section 4, we get an improved result for the linear BSDE, in which the unique existence of the BMO solution can be obtained via a weak condition for the terminal value, i.e. $\xi \in BMO$.

2. Preliminaries

2.1. Notations and definitions

In this subsection we introduce notations and definitions. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t, t \geq 0\}$ satisfying the usual conditions: (i) $\mathscr{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathscr{F}$; (ii) $\mathscr{F}_t = \cap_{s>t}\mathscr{F}_s$ for all $t \geq 0$. Throughout this paper we assume that all the processes equal to *zero* at $t = 0-$. Let $\mathcal{M}_{loc,0}(\mathbb{P})$ be the space of all local martingales $\{M_t, t \geq 0\}$ under the probability measure $\mathbb{P}$, with càdlàg paths and $M_0 = 0$. For simplicity of notations, we write $\mathcal{M}_{loc,0}$ for $\mathcal{M}_{loc,0}(\mathbb{P})$ if there is no danger of confusion. The norm of a $d_1 \times d_2$ matrix y is given by $|y| := \sqrt{trace(yy^{\mathrm{T}})}$. By saying that a vector-valued or matrix-valued function belongs to a function space, we mean all the components belong to that space. Let the terminal time T be a positive number.

The quadratic covariation of $M, N \in \mathcal{M}_{loc,0}$ is defined by

$$[M, N]_t := M_t N_t - \int_0^t M_{s-} dN_s - \int_0^t N_{s-} dM_s, \quad t \geq 0.$$

The notation $[M, M]$ is simplified as $[M]$, which is called the quadratic variation of M. Let ΔM denote the process

$$\Delta M_t = M_t - M_{t-}, \quad t \geq 0.$$

Then, we have

$$\Delta[M, N] = \Delta M \Delta N.$$

Let $p \in [1, \infty]$. For $p \in [1, \infty)$, denote by L^p the space of all $\mathscr{F}_T$-measurable random variables ξ such that

$$\|\xi\|_{L^p} := \left(\mathbb{E}\left[|\xi|^p\right]\right)^{\frac{1}{p}} < \infty,$$

and L^∞ be the space of all essentially bounded and $\mathscr{F}_T$-measurable random variables, equipped with the canonical norm $\|\cdot\|_{L^\infty}$. The space $\mathcal{S}^p$ is the space of all càdlàg adapted processes M such that

$$\|M\|_{\mathcal{S}^p} := \|M_T^*\|_{L^p} < \infty \quad \text{with} \quad M_t^* := \sup_{0 \leq s \leq t} |M_s| \text{ for } t \geq 0.$$

The space $\mathcal{H}^p$ is the space of all processes $M \in \mathcal{M}_{loc,0}$ such that

$$\|M\|_{\mathcal{H}^p} := \left\| [M]_T^{\frac{1}{2}} \right\|_{L^p} < \infty.$$

For the predictable stochastic process H and X, the notation $H \circ X$ stands for the stochastic integral $\int_0^\cdot H_s dX_s$. For any stopping time τ and σ with $\tau < \sigma \leq T$, we define

$${}^\tau X^\sigma := (X - X^\tau)^{\sigma-}$$

with

$$X_t^{\tau-} := X_t \cdot \chi_{[0,\tau)}(t) + X_{\tau-} \cdot \chi_{[\tau,T]}(t),$$

and

$$X_t^\tau := X_t \cdot \chi_{[0,\tau)}(t) + X_\tau \cdot \chi_{[\tau,T]}(t).$$

Definition 2.1. *The space BMO is defined as the set*

$$\left\{ M \in \mathcal{M}_{loc,0} \,\middle|\, \|M\|_{BMO} := \sup_\tau \left\| \left\{ \mathbb{E}\left[|M_T - M_{\tau-}|^2 \,\middle|\, \mathscr{F}_\tau \right] \right\}^{\frac{1}{2}} \right\|_{L^\infty} < \infty \right\}.$$

We say that a random variable ξ lies in BMO if $\mathbb{E}[\xi|\mathscr{F}_\cdot] - \mathbb{E}[\xi] \in BMO$.

The following definition of sliceability is based on [19, page 254].

Definition 2.2. *Let $M = (M_1, \dots, M_n) \in (BMO)^n$ and $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon_l > 0$, $l = 1, \dots, n$. If there is a finite sequence of stopping times satisfying*

$$0 = T_0 < T_1 < \ldots < T_k < T_{k+1} = T$$

such that

$$\left\| {}^{T_i}M_l^{T_{i+1}} \right\|_{BMO} \le \varepsilon_l, \qquad i = 0, \dots, k, \quad l = 1, \dots, n,$$

we say that M is $\vec{\varepsilon}$-sliceable in $(BMO)^n$.

Definition 2.3. *Let $M \in \mathcal{M}_{loc,0}$ and $\mathbb{L}(M)$ be the totality of all predictable processes which are integrable with respect to M. Write*

$$\mathcal{L}(M) = \{H \circ M : H \in \mathbb{L}(M)\}.$$

If $\mathcal{L}(M) = \mathcal{M}_{loc,0}$, we say that M has the strong property of predictable representation.

Definition 2.4. *Define*

$$\Gamma(M) := \left\{\mathbb{P}' : \ \mathbb{P}' \text{ is a probability measure on } \mathscr{F} \text{ and } M \in \mathcal{M}_{loc,0}(\mathbb{P}')\right\},$$

Denote by $\Gamma_e(M)$ the set of extreme points of $\Gamma(M)$, i.e.,

$$\Gamma_e(M) := \Big\{\mathbb{P}' \in \Gamma(M) \colon \text{if } \mathbb{P}' = a\mathbb{P}_1 + (1-a)\mathbb{P}_2,\ \mathbb{P}_1, \mathbb{P}_2 \in \Gamma(M),\ a \in (0,1) \\ \text{then } \mathbb{P}' = \mathbb{P}^1 = \mathbb{P}^2\Big\}.$$

2.2. Some inequalities and lemmas

In this subsection, we recall some well-known inequalities and lemmas, which can be found in e.g. He et al. [14], Kazamaki [15] and Protter [19].

Lemma 2.1 (Doob's inequality)**.** *Let M be a positive submartingale. For $p \in (1, \infty)$ with q conjugate to p, we have*

$$\|M\|_{\mathcal{S}^p} \le q \|M_T\|_{L^p}.$$

Lemma 2.2 (BDG inequality)**.** *For $p \in [1, \infty)$, there exist two constants $C_p > c_p > 0$ such that*

$$C_p^{-1} \|M\|_{\mathcal{H}^p} \le \|M\|_{\mathcal{S}^p} \le c_p^{-1} \|M\|_{\mathcal{H}^p}, \qquad \forall M \in \mathcal{H}^p.$$

Lemma 2.3 (Kunita-Watanabe inequality)**.** *Let H, K be measurable processes and $X, Y \in \mathcal{M}_{loc,0}$. Then, one has almost surely*

$$\int_0^T |H_s||K_s||d[X,Y]_s| \le \left(\int_0^T H_s^2 d[X]_s\right)^{\frac{1}{2}} \left(\int_0^T K_s^2 d[Y]_s\right)^{\frac{1}{2}}.$$

Lemma 2.4 (Fefferman's inequality)**.** *Let $M, N \in \mathcal{M}_{loc,0}$, U be an optional process and τ be a stopping time. We have*

$$\mathbb{E}\left[\int_\tau^T |U_s||d[M,N]_s| \,\middle|\, \mathscr{F}_\tau\right] \le \sqrt{2}\mathbb{E}\left[\left(\int_\tau^T U_s^2 d[M]_s\right)^{\frac{1}{2}} \,\middle|\, \mathscr{F}_\tau\right] \|N\|_{BMO}.$$

The dual space of continuous linear functional on $\mathcal{H}^p$ is $\mathcal{H}^q$, where $1/p + 1/q = 1$, $1 < p < \infty$. For the dual space of $\mathcal{H}^1$, we have the following lemma.

Lemma 2.5. *For a fixed $N \in BMO$, define $\varphi_N(M) = \mathbb{E}\left[[N,M]_T\right]$ for $M \in \mathcal{H}^1$. Then $N \mapsto \varphi_N$ is a one to one linear mapping from BMO onto $\left(\mathcal{H}^1\right)^*$ and $\frac{1}{\sqrt{2}}\|\varphi_N\| \le \|N\|_{BMO} \le \sqrt{5}\|\varphi_N\|$.*

Lemma 2.6. *Let $M, N \in \mathcal{H}^2$. Define*

$$Int(M) := \left\{H \circ M \in \mathcal{L}(M) : \ \|H \circ M\|_{\mathcal{H}^2} < \infty\right\}.$$

Let $L = Z \circ M$ be the projection of N onto $Int(M)$. We have that $M^\perp := N - L$ is orthogonal to $Int(M)$, i.e., for any $H \circ M \in Int(M)$, $[M^\perp, H \circ M]$ is a martingale.

Lemma 2.7. *The following two assertions are equivalent.*

(i) M has the strong property of predicable representation;
(ii) $\mathbb{P} \in \Gamma_e(M)$.

2.3. Some important lemmas

We have the following Lemmas 2.8 and 2.9 as the discontinuous counterparts of the results in Delbaen and Tang [9]. They play an essential role in the proof of our main results.

Lemma 2.8. *Let $p \in [1, \infty)$. If $X \in \mathcal{S}^p$ and $M \in BMO$, we have*

$$\|X_- \circ M\|_{\mathcal{H}^p} \le \sqrt{2}\|X\|_{\mathcal{S}^p}\|M\|_{BMO}.$$

Proof. (i) The case $p \in (1,\infty)$. From Fefferman's and Hölder's inequalities, we have for any $N \in \mathcal{H}^q$,

$$\begin{aligned} |\mathbb{E}\{[X_- \circ M, N]_T\}| &\leq \mathbb{E}\{|[X_- \circ N, M]_T|\} \\ &\leq \sqrt{2}\|X_- \circ N\|_{\mathcal{H}^1}\|M\|_{BMO} \\ &\leq \sqrt{2}\|X\|_{\mathcal{S}^p}\|N\|_{\mathcal{H}^q}\|M\|_{BMO}. \end{aligned}$$

Therefore,

$$\|X_- \circ M\|_{H^p} = \sup_{N\in\mathcal{H}^q} \frac{|\mathbb{E}\{[X_- \circ M, N]_T\}|}{\|N\|_{\mathcal{H}^q}} \leq \sqrt{2}\|X\|_{\mathcal{S}^p}\|M\|_{BMO}.$$

(ii) The case $p = 1$. We have

$$\begin{aligned} \int_0^T X_{s-}^2 d[M]_s &\leq X_T^* \int_0^T X_{s-}^* d[M]_s \\ &= X_T^* \left(X_T^*[M]_T - \int_0^T [M]_{s-} dX_s^* - [[M], X^*]_T \right). \end{aligned}$$

Since $[M]$ and X^* are nondecreasing processes, we have

$$[[M], X^*]_T = \sum_{0<s\leq T} \Delta[M]_s \Delta X_s^* \geq 0.$$

It holds that

$$\begin{aligned} \int_0^T X_{s-}^2 d[M]_s &\leq X_T^* \left(X_T^*[M]_T - \int_0^T [M]_{s-} dX_s^* \right) \\ &= X_T^* \left(\int_0^T ([M]_T - [M]_{s-})\, dX_s^* + X_0^*[M]_T \right). \end{aligned}$$

Therefore,

$$\begin{aligned} &\mathbb{E}\left[\left(\int_0^T X_{s-}^2 d[M]_s \right)^{\frac{1}{2}} \right] \\ \leq\ & \mathbb{E}\left[(X_T^*)^{\frac{1}{2}} \left(\int_0^T ([M]_T - [M]_{s-})\, dX_s^* + X_0^*[M]_T \right)^{\frac{1}{2}} \right] \\ \leq\ & \|X\|_{\mathcal{S}^1}^{\frac{1}{2}} \left\{ \mathbb{E}\left[\int_0^T \mathbb{E}\left[[M]_T - [M]_{s-} | \mathscr{F}_s\right] dX_s^* \right] + X_0^*\mathbb{E}[[M]_T] \right\}^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\le \|X\|_{\mathcal{S}^1}^{\frac{1}{2}} \left\{ \mathbb{E}\left[\int_0^T \|M\|_{BMO}^2 \, dX_s^* \right] + X_0^* \|M\|_{BMO}^2 \right\}^{\frac{1}{2}} \\
&\le \|X\|_{\mathcal{S}^1}^{\frac{1}{2}} \|M\|_{BMO} \left(\mathbb{E}\left[X_T^*\right]\right)^{\frac{1}{2}} \le \sqrt{2}\|X\|_{\mathcal{S}^1}\|M\|_{BMO}.
\end{aligned}
$$

This completes the proof. □

Lemma 2.9. *Let $p \in [1, \infty)$. If $X \in \mathcal{H}^p$ and $M \in BMO$, we have*

$$
\left\| \int_0^T |d[M, X]_s| \right\|_{L^p} \le \sqrt{2} p \|X\|_{\mathcal{H}^p} \|M\|_{BMO}.
$$

Proof. For the case $p = 1$, it is immediate from Fefferman's inequality to get the desired results. In what follows, we consider the case $p \in (1, \infty)$. Take $\xi \in L^q$ with $1/p + 1/q = 1$. Write $Y_t := \mathbb{E}[|\xi||\mathscr{F}_t]$ for $t \in [0, T]$. According to Fefferman's inequality, it follows

$$
\begin{aligned}
\left| \mathbb{E}\left[\left(\int_0^T |d[M, X]_s| \right) \xi \right] \right| &\le \mathbb{E}\left[\int_0^T Y_s |d[X, M]_s| \right] \\
&\le \sqrt{2}\mathbb{E}\left[\left(\int_0^T Y_s^2 d[X]_s \right)^{\frac{1}{2}} \right] \|M\|_{BMO}.
\end{aligned}
$$

In view of Hölder's inequality and Doob's inequality, we have

$$
\begin{aligned}
\left| \mathbb{E}\left[\left(\int_0^T |d[M, X]_s| \right) \xi \right] \right| &\le \sqrt{2}\|X\|_{\mathcal{H}^p}\|M\|_{BMO}\|Y\|_{\mathcal{S}^q} \\
&\le \sqrt{2}p\|X\|_{\mathcal{H}^p}\|\xi\|_{L^q}\|M\|_{BMO}.
\end{aligned}
$$

The proof is complete. □

3. Nonlinear BSDEs with jumps

Throughout this section, N_1, N_2, N_3 and M are supposed to lie in $\mathcal{M}_{loc,0}$, the terminal value ξ is an $\mathbb{R}^n$-valued $\mathscr{F}_T$-measurable random variable, J is an $\mathbb{R}^n$-valued $\{\mathscr{F}_t,\ 0 \le t \le T\}$-adapted càdlàg process, and the $\mathbb{R}^n$-valued random functions f and g are defined on $\Omega \times [0, T] \times \mathbb{R}^n$ and $\Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$, respectively. For each $y, z \in \mathbb{R}^n$, $f(\cdot, \cdot, y)$ and $g(\cdot, \cdot, y, z)$ are adapted. Moreover, we assume that there are three adapted processes α, β, γ such that for each $(\omega, t) \in \Omega \times [0, T]$, it holds

$$
f(t, 0) = 0, \quad |f(t, y_1) - f(t, y_2)| \le \alpha(t)|y_1 - y_2|
$$

for any $(y_1, y_2) \in (\mathbb{R}^n)^2$, and

$$g(t, 0, 0) = 0, \quad |g(t, y_1, z_1) - g(t, y_2, z_2)| \le \beta(t)|y_1 - y_2| + \gamma(t)|z_1 - z_2|$$

for any $(y_1, y_2, z_1, z_2) \in (\mathbb{R}^n)^4$.

3.1. The $\mathcal{S}^2 \times (\mathcal{H}^2)^2$ solution

Let us first study the case of $p = 2$. The triple of processes $(Y, Z \circ M, M^\perp)$ are called a solution of BSDE (1) if (i) they satisfy the equation (1) and are $\{\mathscr{F}_t,\ 0 \le t \le T\}$-adapted, (ii) Z is a predictable process, and (iii) M and $M^\perp$ are strongly orthogonal (i.e., $[M, M^\perp]$ is a martingale). We have the following existence and uniqueness result for $\mathcal{S}^2 \times (\mathcal{H}^2)^2$ solution.

Theorem 3.1. *Let $M \in \mathcal{H}^2$. Assume that $(N_2, \beta^{\frac{1}{2}} \circ M, \gamma \circ N_3)$ is $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$-sliceable in $(BMO)^3$ and $(\alpha \circ N_1, \beta^{\frac{1}{2}} \circ N_3)$ belongs to $(BMO)^2$ such that*

$$\rho_2 := \overline{C}_2 \max\left\{2\sqrt{2}\,\varepsilon_3, 4\left(\|\alpha \circ N_1\|_{BMO}\,\varepsilon_1 + \varepsilon_2 \left\|\beta^{\frac{1}{2}} \circ N_3\right\|_{BMO}\right)\right\} < 1,$$

where $\overline{C}_2 := 6C_2 + 2$, and C_2 is the constant in BDG inequality for $p = 2$.

Then for any $(\xi, J) \in L^2 \times \mathcal{S}^2$, BSDE (1) has a unique solution $(Y, Z \circ M, M^\perp) \in \mathcal{S}^2 \times (\mathcal{H}^2)^2$ such that

$$\|Y\|_{\mathcal{S}^2} + \|Z \circ M\|_{\mathcal{H}^2} + \|M^\perp\|_{\mathcal{H}^2} \le K_2\left(\|\xi\|_{L^2} + \|J\|_{\mathcal{S}^2}\right),$$

where K_2 is a positive constant independent of (ξ, J).

Proof. For any $y \in \mathcal{S}^2$ and $z \circ M \in \mathcal{H}^2$, we consider the following BSDE:

$$\begin{aligned} Y_t &= \xi + J_T - J_t + \int_{t+}^T f(s, y_{s-}) d[N_1, N_2]_s \\ &\quad + \int_{t+}^T g(s, y_{s-}, z_s) d[N_3, M]_s - \int_{t+}^T Z_s dM_s - \int_{t+}^T dM_s^\perp. \end{aligned} \tag{2}$$

Define

$$\begin{aligned} F_t &:= \mathbb{E}\Big[\xi + J_T + \int_0^T f(s, y_{s-}) d[N_1, N_2]_s \\ &\qquad + \int_0^T g(s, y_{s-}, z_s) d[N_3, M]_s \,\Big|\, \mathscr{F}_t\Big]. \end{aligned} \tag{3}$$

By Doob's inequality, we have

$$\begin{aligned}\|F\|_{\mathcal{S}^2} &\leq 2\,\|\xi+J_T\|_{L^2}+2\left\|\int_0^T f(s,y_{s-})d[N_1,N_2]_s\right\|_{L^2}\\ &\quad +2\left\|\int_0^T g(s,y_{s-},z_s)d[N_3,M]_s\right\|_{L^2}.\end{aligned}$$

According to Lemmas 2.8 and 2.9, we have

$$\begin{aligned}\left\|\int_0^T f(s,y_{s-})\,d[N_1,N_2]_s\right\|_{L^2} &\leq \left\|\int_0^T \alpha(s)|y_{s-}|\,|d[N_1,N_2]_s|\right\|_{L^2}\\ &= \left\|\int_0^T |d[\alpha\circ N_1,|y_-|\circ N_2]_s|\right\|_{L^2}\\ &\leq 2\sqrt{2}\,\|y_-^*\circ N_2\|_{\mathcal{H}^2}\|\alpha\circ N_1\|_{BMO}\\ &\leq 4\|y\|_{\mathcal{S}^2}\|N_2\|_{BMO}\|\alpha\circ N_1\|_{BMO}\end{aligned}$$

and

$$\begin{aligned}\left\|\int_0^T g(s,y_{s-},z_s)\,d[N_3,M]_s\right\|_{L^2} &\leq \left\|\int_0^T |d[\beta^{\frac{1}{2}}\circ N_3,|y_-|\circ\beta^{\frac{1}{2}}\circ M]_s|\right\|_{L^2}\\ &\quad +\left\|\int_0^T |d[\gamma\circ N_3,|z|\circ M]_s|\right\|_{L^2}\\ &\leq 4\|y\|_{\mathcal{S}^2}\|\beta^{\frac{1}{2}}\circ M\|_{BMO}\|\beta^{\frac{1}{2}}\circ N_3\|_{BMO}\\ &\quad +2\sqrt{2}\|z\circ M\|_{\mathcal{H}^2}\|\gamma\circ N_3\|_{BMO}.\end{aligned}$$

Therefore, $F\in\mathcal{S}^2$. Then Lemma 2.6 implies that there exist a predictable process Z and a martingale $M^\perp$ orthogonal to $Int(M)$, such that

$$F_t = F_0+\int_0^t Z_s\,dM_s+M_t^\perp.$$

It holds that

$$\begin{aligned}&F_t+\int_{t+}^T Z_s\,dM_s+\int_{t+}^T dM_s^\perp = F_T\\ =\;& \xi+J_T+\int_0^T f(s,y_{s-})\,d[N_1,N_2]_s+\int_0^T g(s,y_{s-},z_s)\,d[N_3,M]_s.\end{aligned}$$

We define

$$Y_t := F_t - J_t - \int_0^t f(s, y_{s-})\, d[N_1, N_2]_s - \int_0^t g(s, y_{s-}, z_s)\, d[N_3, M]_s.$$

Then the triple $(Y, Z \circ M, M^{\perp})$ is a solution of BSDE (2).

Note that

$$\begin{aligned} Y_t &= \mathbb{E}\Big[\xi + J_T - J_t + \int_{t+}^T f(s, y_{s-}) d[N_1, N_2]_s \\ &\qquad + \int_{t+}^T g(s, y_{s-}, z_s) d[N_3, M]_s \,\Big|\, \mathscr{F}_t\Big]. \end{aligned}$$

In view of Doob's inequality, we have

$$\begin{aligned} \|Y\|_{\mathcal{S}^2} &\leq 2\, \|\xi + J_T\|_{L^2} + \|J\|_{\mathcal{S}^2} \\ &\quad +2 \left\| \int_0^T |f(s, y_{s-})|\, d[N_1, N_2]_s \right\|_{L^2} \\ &\quad +2 \left\| \int_0^T |g(s, y_{s-}, z_s)|\, d[N_3, M]_s \right\|_{L^2} \\ &\leq \|J\|_{\mathcal{S}^2} + 2\|\xi + J_T\|_{L^2} \\ &\quad +8\|y\|_{\mathcal{S}^2} \|N_2\|_{BMO} \|\alpha \circ N_1\|_{BMO} \\ &\quad +8\|y\|_{\mathcal{S}^2} \|\beta^{\frac{1}{2}} \circ M\|_{BMO} \|\beta^{\frac{1}{2}} \circ N_3\|_{BMO} \\ &\quad +4\sqrt{2}\|z \circ M\|_{\mathcal{H}^2} \|\gamma \circ N_3\|_{BMO}. \end{aligned}$$

Note that

$$\begin{aligned} \int_{t+}^T Z_s\, dM_s + \int_{t+}^T dM_s^{\perp} &= -Y_t + \xi + J_T - J_t \\ &\quad + \int_{t+}^T f(s, y_{s-})\, d[N_1, N_2]_s \\ &\quad + \int_{t+}^T g(s, y_{s-}, z_s)\, d[N_3, M]_s. \end{aligned}$$

Since $M^{\perp}$ is orthogonal to $Int(M)$, then $[Z \circ M, M^{\perp}]$ is a martingale. We have

$$\|Z \circ M\|_{\mathcal{H}^2}^2 = \mathbb{E}\{[Z \circ M]_T\}$$

$$\begin{aligned}
&\leq \mathbb{E}\left\{[Z\circ M]_T+[M^{\perp}]_T\right\}\\
&= \mathbb{E}\left\{[Z\circ M]_T+[M^{\perp}]_T+2[Z\circ M,M^{\perp}]_T\right\}\\
&= \mathbb{E}\left\{[Z\circ M+M^{\perp}]_T\right\}\\
&= \|Z\circ M+M^{\perp}\|^2_{\mathcal{H}^2}. \qquad (4)
\end{aligned}$$

Using BDG inequality, we have

$$\begin{aligned}
\|Z\circ M\|_{\mathcal{H}^2} &\leq C_2\left\|Z\circ M+M^{\perp}\right\|_{\mathcal{S}^2}\\
&\leq 2C_2\left\|\int_{t+}^T Z_s\,dM_s+\int_{t+}^T dM_s^{\perp}\right\|_{\mathcal{S}^2}\\
&\leq 2C_2\Big(\|Y\|_{\mathcal{S}^2}+\|\xi+J_T\|_{L^2}+\|J\|_{\mathcal{S}^2}\\
&\qquad +4\|y\|_{\mathcal{S}^2}\|N_2\|_{BMO}\|\alpha\circ N_1\|_{BMO}\\
&\qquad +4\|y\|_{\mathcal{S}^2}\|\beta^{\frac{1}{2}}\circ M\|_{BMO}\|\beta^{\frac{1}{2}}\circ N_3\|_{BMO}\\
&\qquad +2\sqrt{2}\|z\circ M\|_{\mathcal{H}^2}\|\gamma\circ N_3\|_{BMO}\Big).
\end{aligned}$$

Combining the above, we have

$$\begin{aligned}
\|Y\|_{\mathcal{S}^2}+\|Z\circ M\|_{\mathcal{H}^2} &\leq \overline{C}_2\|\xi+J_T\|_{L^2}+(1+4C_2)\,\|J\|_{\mathcal{S}^2}\\
&\quad +4\overline{C}_2\,\|\alpha\circ N_1\|_{BMO}\,\|N_2\|_{BMO}\,\|y\|_{\mathcal{S}^2}\\
&\quad +4\overline{C}_2\|\beta^{\frac{1}{2}}\circ M\|_{BMO}\|\beta^{\frac{1}{2}}\circ N_3\|_{BMO}\|y\|_{\mathcal{S}^2}\\
&\quad +2\sqrt{2}\overline{C}_2\,\|\gamma\circ N_3\|_{BMO}\,\|\,z\circ M\,\|_{\mathcal{H}^2}\,. \qquad (5)
\end{aligned}$$

Thus the solution $(Y, Z\circ M, M^{\perp})$ of BSDE (2) lies in $\mathcal{S}^2\times(\mathcal{H}^2)^2$. The uniqueness can be easily proved if one estimates $\|Y^1-Y^2\|_{\mathcal{S}^2}+\|(Z^1-Z^2)\circ M\|_{\mathcal{H}^2}$ by the similar method of (5).

Since the martingale $(N_2, \beta^{\frac{1}{2}}\circ M, \gamma\circ N_3, \beta^{\frac{1}{2}}\circ N_3)$ is $\vec{\varepsilon}$-sliceable in $(BMO)^4$, there is a finite sequence of stopping times $\{T_i,\ i=0,\dots,\widetilde{I}+1\}$ satisfying

$$0=T_0<T_1<T_2<\cdots<T_{\widetilde{I}}<T_{\widetilde{I}+1}=T$$

such that

$$\|N_{2i}\|_{BMO}\leq\varepsilon_1,\quad \|\beta^{\frac{1}{2}}\circ M_i\|_{BMO}\leq\varepsilon_2,$$
$$\|\gamma\circ N_{3i}\|_{BMO}\leq\varepsilon_3,\quad \|\beta^{\frac{1}{2}}\circ N_{3i}\|_{BMO}\leq\varepsilon_4,$$

where

$$N_{1i} := {}^{T_i}N_1^{T_{i+1}}, \quad N_{2i} := {}^{T_i}N_2^{T_{i+1}}, \quad N_{3i} := {}^{T_i}N_3^{T_{i+1}}, \quad M_i := {}^{T_i}M^{T_{i+1}},$$

are defined on $[T_i, T_{i+1}]$ for $i = 0, 1, \cdots, \widetilde{I}$.

Set for $i = 0, \ldots, \widetilde{I}$,

$$\mathcal{S}_i^2 := \mathcal{S}^2[T_i, T_{i+1}] \quad \text{and} \quad \mathcal{H}_i^2 := \mathcal{H}^2[T_i, T_{i+1}]$$

where the space $\mathcal{S}^2[T_i, T_{i+1}]$ (resp. $\mathcal{H}^2[T_i, T_{i+1}]$) consists of all processes of $\mathcal{S}^2$ (resp. $\mathcal{H}^2$) restricted on $[T_i, T_{i+1}]$. Consider the transformation I_i in the Banach space $\mathcal{S}_i^2 \times \mathcal{H}_i^2$: define for $(y, z \circ M) \in \mathcal{S}_i^2 \times \mathcal{H}_i^2$,

$$I_i(y, z \circ M) := (Y^i, Z^i \circ M)$$

as the first two components of the unique adapted solution $(Y, Z \circ M, M^\perp)$ to the following BSDE:

$$\begin{aligned} Y_t &= Y_{T_{i+1}}^{i+1} + (J_{T_{i+1}} - J_t) + \int_{t+}^{T_{i+1}} f(s, y_{s-})\, d[N_{1i}, N_{2i}]_s \\ &\quad + \int_{t+}^{T_{i+1}} g(s, y_{s-}, z_s)\, d[N_{3i}, M_i]_s \\ &\quad - \int_{t+}^{T_{i+1}} Z_s\, dM_{is} - \int_{t+}^{T_{i+1}} dM_s^\perp, \quad t \in [T_i, T_{i+1}] \end{aligned}$$

where $Y_T^{\widetilde{I}+1} := \xi$. Let $(y^k, z^k \circ M) \in \mathcal{S}_i^2 \times \mathcal{H}_i^2$ with $k = 1, 2$. Denote by $(Y^{i,k}, Z^{i,k} \circ M)$ the image $I_i(y^k, z^k \circ M)$ for $k = 1, 2$. Proceeding similarly as before, we have

$$\begin{aligned} & \left\|Y^{i,1} - Y^{i,2}\right\|_{\mathcal{S}_i^2} + \left\|(Z^{i,1} - Z^{i,2}) \circ M\right\|_{\mathcal{H}_i^2} \\ \leq\ & \overline{C}_2 \max\left\{2\sqrt{2}\, \|\gamma \circ N_{3i}\|_{BMO}, C_3\right\} \\ & \times \left[\left\|y^1 - y^2\right\|_{\mathcal{S}_i^2} + \left\|(z^1 - z^2) \circ M\right\|_{\mathcal{H}_i^2}\right] \end{aligned}$$

for the constant

$$C_3 := 4\left(\|\alpha \circ N_{1i}\|_{BMO} \|N_{2i}\|_{BMO} + \left\|\beta^{\frac{1}{2}} \circ M_i\right\|_{BMO} \left\|\beta^{\frac{1}{2}} \circ N_{3i}\right\|_{BMO} \right).$$

Since

$$\|\alpha \circ N_{1i}\|_{BMO} \leq \|\alpha \circ N_1\|_{BMO} \text{ and } \left\|\beta^{\frac{1}{2}} \circ N_{3i}\right\|_{BMO} \leq \left\|\beta^{\frac{1}{2}} \circ N_3\right\|_{BMO},$$

we have

$$\overline{C}_2 \max\left\{2\sqrt{2}\,\|\gamma \circ N_{3i}\|_{BMO}\,, C_3\right\} \leq \rho_2 < 1.$$

Then for any $(y^k, z^k) \in \mathcal{S}_i^2 \times \mathcal{H}_i^2$ with $k = 1, 2$, we have

$$\begin{aligned} & \left\|I_i(y^1, z^1) - I_i(y^2, z^2)\right\|_{\mathcal{S}_i^2 \times \mathcal{H}_i^2} \\ \leq\ & \rho_2 \left[\left\|y^1 - y^2\right\|_{\mathcal{S}_i^2} + \left\|(z^1 - z^2) \circ M\right\|_{\mathcal{H}_i^2}\right]. \end{aligned}$$

Thus I_i is a contraction on $\mathcal{S}_i^2 \times \mathcal{H}_i^2$ for $i = 0, \ldots, \widetilde{I}$. Iteratively in a backward way, the BSDE

$$\begin{aligned} Y_t \ =\ & Y_{T_{i+1}}^{i+1} + (J_{T_{i+1}} - J_t) + \int_{t+}^{T_{i+1}} f(s, Y_{s-})\, d[N_{1i}, N_{2i}]_s \\ & + \int_{t+}^{T_{i+1}} g(s, Y_{s-}, Z_s)\, d[N_{3i}, M_i]_s \\ & - \int_{t+}^{T_{i+1}} Z_s\, dM_{is} - \int_{t+}^{T_{i+1}} dM_s^{\perp}, \quad t \in [T_i, T_{i+1}] \end{aligned}$$

has a unique solution $(Y^i, Z^i \circ M_i, M^{i,\perp}) \in \mathcal{S}_i^2 \times \mathcal{H}_i^2 \times \mathcal{H}_i^2$ for $i = 0, 1, \cdots, \widetilde{I}$. Then, the triple $(Y, Z \circ M, M^{\perp})$ defined by

$$(Y_t, Z_t, M_t^{\perp}) := \sum_{i=0}^{\widetilde{I}} (Y_t^i, Z_t^i, M_t^{i,\perp}) \chi_{[T_i, T_{i+1})}(t), \quad t \in [0, T]$$

is the unique adapted solution to BSDE (1). Moreover, we have

$$\begin{aligned} & (1 - \rho_2)\left(\|Y\|_{\mathcal{S}^2} + \|Z \circ M\|_{\mathcal{H}^2}\right) \\ \leq\ & \overline{C}_2 \|\xi + J_T\|_{L^2} + (4C_2 + 1)\|J\|_{\mathcal{S}^2}. \end{aligned}$$

The proof is complete. □

3.2. The $\mathcal{S}^\infty \times (BMO)^2$ solution

In this subsection, the unique solution $(Y, Z \circ M, M^{\perp})$ of BSDE (1) is further proved to lie in $\mathcal{S}^\infty \times (BMO)^2$ when the data (ξ, J) is essentially bounded and $N_3 = M$ as follows:

$$
\begin{aligned}
Y_t = \xi + J_T - J_t + \int_{t+}^T f(s, Y_{s-}) d[N_1, N_2]_s + \int_{t+}^T g(s, Y_{s-}, Z_s) d[M]_s \\
- \int_{t+}^T Z_s \, dM_s - \int_{t+}^T dM_s^\perp, \qquad\qquad t \in [0, T].
\end{aligned} \tag{6}
$$

Theorem 3.2. *Let $M \in \mathcal{H}^2$. Assume that $(N_2, \beta^{\frac{1}{2}} \circ M, \gamma \circ M)$ is $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$-sliceable in $(BMO)^3$ and $\alpha \circ N_1$ lies in BMO such that*

$$
\rho_2 := (6C_2 + 2) \max \left\{ 2\sqrt{2}\, \varepsilon_3, 4 \left(\|\alpha \circ N_1\|_{BMO}\, \varepsilon_1 + \varepsilon_2^2 \right) \right\} < 1
$$

and

$$
\eta_2 := 4 \left(\varepsilon_1 \|\alpha \circ N_1\|_{BMO} + \varepsilon_2^2 \right) + \frac{\sqrt{2}}{c_2} \varepsilon_3 < 1,
$$

where c_2 and C_2 are the constants in BDG inequality for $p = 2$.

Then for any $(\xi, J) \in L^\infty \times \mathcal{S}^\infty$, BSDE (6) admits a unique adapted solution $(Y, Z \circ M, M^\perp) \in \mathcal{S}^\infty \times (BMO)^2$.

To prove Theorem 3.2, we consider the dual equation of BSDE (6), which is the following $n \times n$ matrix-valued SDE:

$$
\begin{aligned}
S(t) &= \mathcal{I} + \int_{t_0+}^t S(s-) A_s d[N_1, N_2]_s + \int_{t_0+}^t S(s-) B_s d[M]_s \\
&\quad + \int_{t_0+}^t S(s-) D_s dM_s,
\end{aligned} \tag{7}
$$

where $\mathcal{I}$ is the $n \times n$ identity matrix and $t_0 \in [0, T]$ is a fixed number. In the following Corollary, we prove that the SDE (7) admits a unique solution $S(\cdot) \in \mathcal{S}^p$.

Corollary 3.1. *Let $p \in [1, \infty)$. Assume that*

(i) The $R^{n \times n}$-valued processes A, B and D are bounded by the real valued nonnegative adapted processes α, β and γ, respectively;

(ii) $(N_2, \beta^{\frac{1}{2}} \circ M, \gamma \circ M)$ is $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$-sliceable in $(BMO)^3$ and $\alpha \circ N_1$ is in BMO such that

$$
\eta_p := 2p \left(\varepsilon_1 \|\alpha \circ N_1\|_{BMO} + \varepsilon_2^2 \right) + \frac{\sqrt{2}}{c_p} \varepsilon_3 < 1.
$$

Then SDE (7) *admits a unique solution* $S(\cdot)$ *such that*

$$\mathbb{E}\left[\sup_{t\in[t_0,T]}|S(t)|^p\middle|\mathscr{F}_{t_0}\right]^{\frac{1}{p}}\leq B_p,$$

where B_p *is a positive constant independent of* t_0*.*

Proof of Corollary 3.1. Firstly, we use the contraction mapping principle to prove the existence and uniqueness of the solution. Note that the martingale $(N_2,\beta^{\frac{1}{2}}\circ M,\gamma\circ M)$ is $\vec{\varepsilon}$-sliceable in $(BMO)^3$ with the corresponding finite sequence of stopping times $\{T_i,\ i=0,\ldots,\widetilde{I}+1\}$. We consider the following map I_i in the Banach space $\mathcal{S}^p_i$, $i=0,\ldots,\widetilde{I}$. For $S\in\mathcal{S}^p_i$,

$$\begin{aligned}I_i(S)_t &:= S^{i-1}(T_i)+\int_{T_i+}^t S(s-)A_s d[N_{1i},N_{2i}]_s+\int_{T_i+}^t S(s-)B_s d[M_i]_s\\&\quad+\int_{T_i+}^t S(s-)D_s dM_{is},\end{aligned}$$

where $S^{-1}=\mathcal{I}$. According to Lemmas 2.8 and 2.9, it follows

$$\begin{aligned}&\left\|\int_{T_i+}^t S(s-)A_s d[N_{1i},N_{2i}]_s+\int_{T_i+}^t S(s-)B_s d[M_i]_s\right\|_{\mathcal{S}^p_i}\\ \leq\ & 2p\left(\|N_{2i}\|_{BMO}\,\|\alpha\circ N_{1i}\|_{BMO}+\|\beta^{\frac{1}{2}}\circ M_i\|^2_{BMO}\right)\|S\|_{\mathcal{S}^p_i}.\end{aligned}$$

According to the BDG inequality, we have

$$\left\|\int_{T_i+}^t S(s-)D_s dM_{is}\right\|_{\mathcal{S}^p_i}\leq\frac{\sqrt{2}}{c_p}\|\gamma\circ M_i\|_{BMO}\|S\|_{\mathcal{S}^p_i}.$$

It follows

$$\left\|I_i(S^1)-I_i(S^2)\right\|_{\mathcal{S}^p_i}\leq\eta_p\left\|S^1-S^2\right\|_{\mathcal{S}^p_i}.$$

Thus the map I_i is a contraction map and satisfies the following estimate:

$$\|I_i(S)\|_{\mathcal{S}^p_i}\leq\|S^{i-1}(T_i)\|_{L^p_i}+\eta_p\|S\|_{\mathcal{S}^p_i}.$$

Therefore, we get that the stochastic equation

$$S(t)\ =\ S^{i-1}(T_i)+\int_{T_i+}^t S(s-)A_s d[N_{1i},N_{2i}]_s$$

$$+\int_{T_i+}^{t} S(s-)B_s d[N_{3i}, M_i]_s + \int_{T_i+}^{t} S(s-)D_s dN_{3is},$$

has a unique solution $S^i(\cdot)$ in $\mathcal{S}_i^p$ for $i = 0, \ldots, \widetilde{I}$. Then, the process

$$S(t) := \sum_{i=0}^{\widetilde{I}} S^i(t)\chi_{[T_i,T_{i+1})}(t),$$

lies in $\mathcal{S}^p$ and is the unique solution to equation (7). Moreover, we have

$$\mathbb{E}\left[\sup_{t\in[t_0,T]} |S(t)|^p \middle| \mathscr{F}_{t_0}\right]^{\frac{1}{p}} \le B_p$$

with $B_p := n(1-\eta_p)^{-1}$. The proof is complete. □

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2. In view of assumption (i), Theorem 3.1 tells us that BSDE (6) has a unique adapted solution $(Y, Z\circ M, M^{\perp}) \in \mathcal{S}^2 \times (\mathcal{H}^2)^2$. The rest of the proof is divided into the following two steps.

Step 1. We show that $Y \in \mathcal{S}^{\infty}$. Note that BSDE (6) can be written into the following form:

$$\begin{aligned} Y_t &= \xi + J_T - J_t + \int_{t+}^{T} A_s Y_{s-} d[N_1, N_2]_s \\ &\quad + \int_{t+}^{T} (B_s Y_{s-} + D_s Z_s)\, d[M]_s - \int_{t+}^{T} Z_s dM_s, \end{aligned} \tag{8}$$

where the matrix-valued processes A, B, and D are defined by

$$\begin{cases} (A_s)_{ij} &= \dfrac{f^i(s,\bar{Y}^j_{s-}) - f^i(s,\bar{Y}^{j+1}_{s-})}{Y^j_{s-}} \cdot \chi_{\{Y^j_{s-}\neq 0\}} \\ (B_s)_{ij} &= \dfrac{g^i(s,\bar{Y}^j_{s-},0) - g^i(s,\bar{Y}^{j+1}_{s-},0)}{Y^j_{s-}} \cdot \chi_{\{Y^j_{s-}\neq 0\}} \\ (D_s)_{ij} &= \dfrac{g^i(s,Y_{s-},\bar{Z}^j_s) - g^i(s,Y_{s-},\bar{Z}^{j+1}_s)}{Z^j_s} \cdot \chi_{\{Z^j_s\neq 0\}} \end{cases}$$

with $\bar{Y}^j := (0, \ldots, 0, Y^j, Y^{j+1}, \ldots, Y^n)^{\mathrm{T}}$ and $\bar{Z}^j := (0, \ldots, 0, Z^j, Z^{j+1}, \ldots, Z^n)^{\mathrm{T}}$. By the Lipschitz assumption on the coefficients, we know that

$$|A_s| \leq \alpha(s), \quad |B_s| \leq \beta(s), \quad |D_s| \leq \gamma(s).$$

Then by the assumption (i) and Corollary 3.1, we know that SDE (7) has a unique solution $S(\cdot) \in \mathcal{S}^2$. Fix an arbitrary $t_0 \in [0, T]$. Applying the Itô's formula for $S(t)(Y_t + J_t)$, we have

$$\begin{aligned} Y_{t_0} + J_{t_0} &= \mathbb{E}\left[S(T)(\xi + J_T) \,\middle|\, \mathscr{F}_{t_0} \right] \\ &\quad - \mathbb{E}\left[\int_{t_0+}^T S(s-) A_s J_{s-} \, d[N_1, N_2]_s \,\middle|\, \mathscr{F}_{t_0} \right] \\ &\quad - \mathbb{E}\left[\int_{t_0+}^T S(s-) B_s J_{s-} \, d[M]_s \,\middle|\, \mathscr{F}_{t_0} \right]. \end{aligned}$$

It follows

$$\begin{aligned} |Y_{t_0}| &\leq |J_{t_0}| + \mathbb{E}\left[|S(T)| \, |\xi + J_T| \, |\mathscr{F}_{t_0} \right] \\ &\quad + \|J\|_{\mathcal{S}^\infty} \mathbb{E}\left[\left(\sup_{s \in [t_0, T]} |S(s)| \right) \int_{t_0+}^T |\alpha_s d[N_1, N_2]_s| \,\middle|\, \mathscr{F}_{t_0} \right] \\ &\quad + \|J\|_{\mathcal{S}^\infty} \mathbb{E}\left[\left(\sup_{s \in [t_0, T]} |S(s)| \right) \int_{t_0+}^T \beta_s d[M]_s \,\middle|\, \mathscr{F}_{t_0} \right]. \end{aligned}$$

According to Hölder's inequality and Kunita-Watanabe inequality, we have

$$\begin{aligned} |Y_{t_0}| &\leq \|J\|_{\mathcal{S}^\infty} + B_2 \left(\|\xi\|_{L^\infty} + \|J\|_{\mathcal{S}^\infty} \right) \\ &\quad + B_2 \|J\|_{\mathcal{S}^\infty} \left\{ \mathbb{E}\left[[\alpha \circ N_1] \big|_{t_0}^T + [N_2] \big|_{t_0}^T \,\middle|\, \mathscr{F}_{t_0} \right] \right\}^{\frac{1}{2}} \\ &\quad + B_2 \|J\|_{\mathcal{S}^\infty} \left\{ \mathbb{E}\left[[\beta^{\frac{1}{2}} \circ M] \big|_{t_0}^T \,\middle|\, \mathscr{F}_{t_0} \right] \right\}^{1/2}, \end{aligned}$$

where B_2 is the constant in Corollary 3.1 for $p = 2$. Consequently, we have

$$\begin{aligned} \|Y\|_{\mathcal{S}^\infty} &\leq \|J\|_{\mathcal{S}^\infty} + B_2 \left(\|\xi\|_{L^\infty} + \|J\|_{\mathcal{S}^\infty} \right) \\ &\quad + B_2 \|J\|_{\mathcal{S}^\infty} \left(\|\alpha \circ N_1\|_{BMO} + \|N_2\|_{BMO} + \left\| \beta^{\frac{1}{2}} \circ M \right\|_{BMO} \right). \end{aligned}$$

Step 2. We show that $Z \circ M \in BMO$. To simplify the exposition, we set

$$C_{YJ} := \|Y\|_{\mathcal{S}^\infty} + \|J\|_{\mathcal{S}^\infty}.$$

In view of BSDE (8), using Itô's formula and standard arguments, we have

$$\begin{aligned}
[Z \circ M + M^{\perp}]\big|_{\sigma-}^{T} &= |\xi + J_T|^2 - |Y_{\sigma-} + J_{\sigma-}|^2 \\
&\quad +2\int_\sigma^T (Y_{t-} + J_{t-})^{\mathrm{T}} A_t Y_t d[N_1, N_2]_t \\
&\quad +2\int_\sigma^T (Y_{t-} + J_{t-})^{\mathrm{T}} (B_t Y_t + D_t Z_t) d[M]_t \\
&\quad -2\int_\sigma^T (Y_{t-} + J_{t-})^{\mathrm{T}} d(Z \circ M + M^{\perp})_t,
\end{aligned}$$

for any stopping time $\sigma \le T$. It follows that

$$\begin{aligned}
\mathbb{E}\left[[Z \circ M]\big|_{\sigma-}^{T} | \mathscr{F}_\sigma\right] &\le \mathbb{E}\left[[Z \circ M + M^{\perp}]\big|_{\sigma-}^{T} | \mathscr{F}_\sigma\right] \\
&\le C_{YJ}^2 + 2C_{YJ}\mathbb{E}\left[\int_\sigma^T \alpha_s |Y_s| |d[N_1, N_2]_s| \,\middle|\, \mathscr{F}_\sigma\right] \\
&\quad +2C_{YJ}\mathbb{E}\left[\int_\sigma^T \beta_s |Y_s| d[M]_s \,\middle|\, \mathscr{F}_\sigma\right] \\
&\quad +2C_{YJ}\mathbb{E}\left[\int_\sigma^T \gamma_s |Z_s| d[M]_s \,\middle|\, \mathscr{F}_\sigma\right].
\end{aligned}$$

According to Kunita-Watanabe inequality and Young's inequality, we have

$$\begin{aligned}
\mathbb{E}\left[[Z \circ M]\big|_{\sigma-}^{T} | \mathscr{F}_\sigma\right] &\le C_{YJ}^2 + 2C_{YJ}^2\mathbb{E}\left[\left([\alpha \circ N_1]\big|_\sigma^T\right)^{\frac{1}{2}} \left([N_2]\big|_\sigma^T\right)^{\frac{1}{2}} \,\middle|\, \mathscr{F}_\sigma\right] \\
&\quad +2C_{YJ}^2\mathbb{E}\left[[\beta^{\frac{1}{2}} \circ M]\big|_\sigma^T \,\middle|\, \mathscr{F}_\sigma\right] \\
&\quad +2C_{YJ}\mathbb{E}\left[\left([\gamma \circ M]\big|_\sigma^T\right)^{\frac{1}{2}} \left([Z \circ M]\big|_\sigma^T\right)^{\frac{1}{2}} \,\middle|\, \mathscr{F}_\sigma\right] \\
&\le C_{YJ}^2 \left(1 + \|\alpha \circ N_1\|_{BMO}^2 + \|N_2\|_{BMO}^2\right) \\
&\quad +2C_{YJ}^2 \left(\|\gamma \circ M\|_{BMO}^2 + \|\beta^{\frac{1}{2}} \circ M\|_{BMO}^2\right) \\
&\quad +\frac{1}{2}\mathbb{E}\left[[Z \circ M]\big|_{\sigma-}^{T} \,\middle|\, \mathscr{F}_\sigma\right].
\end{aligned}$$

It holds that

$$\begin{aligned}\|Z\circ M\|_{BMO} &= \sup_{\sigma}\mathbb{E}\left[[Z\circ M]\big|_{\sigma-}^{T}|\mathscr{F}_\sigma\right]\\ &\le 2C_{YJ}^2\left(1+\|\alpha\circ N_1\|_{BMO}^2+\|N_2\|_{BMO}^2\right)\\ &\quad +4C_{YJ}^2\left(\|\beta^{\frac{1}{2}}\circ M\|_{BMO}^2+\|\gamma\circ M\|_{BMO}^2\right).\end{aligned}$$

Thus $Z\circ M\in BMO$. This completes the proof. $\square$

3.3. The $\mathcal{S}^p\times\mathcal{H}^p$ solution with $p\in(1,\infty)$

In Section 3.1, the inequality

$$\|Z\circ M\|_{\mathcal{H}^2}\le\|Z\circ M+M^\perp\|_{\mathcal{H}^2}$$

is shown to be important in the construction of the contraction map. In the general case of $p\in(1,\infty)$, for continuous M, we have that

$$[Z\circ M,M^\perp]=\langle Z\circ M,M^\perp\rangle=0,$$

and the norm $\|Z\circ M\|_{\mathcal{H}^p}$ can be estimated via the inequality

$$\|Z\circ M\|_{\mathcal{H}^p}\le\|Z\circ M+M^\perp\|_{\mathcal{H}^p},$$

which has been well addressed by Delbean and Tang [9] in 2010. However, for discontinuous M, both norms $\|Z\circ M\|_{\mathcal{H}^p}$ and $\|M^\perp\|_{\mathcal{H}^p}$ could not be separately estimated as in (4), and we can only get that $[Z\circ M,M^\perp]$ is a martingale and

$$\|Z\circ M+M^\perp\|_{\mathcal{H}^p}<\infty.$$

This difference turns out to be a stumbling block to the definition of the contraction map in our general case. To sidestep the trouble, we assume that the strong predictable representation property holds and consider the BSDEs of the following form:

$$\begin{aligned}Y_t &= \xi+J_T-J_t+\int_{t+}^T f(s,Y_{s-})d[N_1,N_2]_s+\int_{t+}^T g(s,Y_{s-},Z_s)d[N_3,M]_s\\ &\quad -\int_{t+}^T Z_s\,dM_s,\qquad t\in[0,T].\end{aligned}\tag{9}$$

By a solution $(Y,Z\circ M)$ to BSDE (9), we mean that (i) $(Y,Z\circ M)$ satisfies the equation (9) and (ii) $(Y,Z\circ M)$ is adapted and Z is predictable.

Theorem 3.3. *Let $\mathbb{P} \in \Gamma_e(M)$ and $p \in (1,\infty)$ with q being its conjugate number. Assume that $(N_2, \beta^{\frac{1}{2}} \circ M, \gamma \circ N_3)$ is $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$-sliceable in $(BMO)^3$ and $(\alpha \circ N_1, \beta^{\frac{1}{2}} \circ N_3)$ is in $(BMO)^2$ such that*

$$\rho_p := \overline{C}_p \max\left\{\sqrt{2}p\varepsilon_3, 2p\left(\varepsilon_1 \left\|\alpha \circ N_1\right\|_{BMO} + \varepsilon_2 \left\|\beta^{\frac{1}{2}} \circ N_3\right\|_{BMO}\right)\right\} < 1,$$

where $\overline{C}_p := 2(q+1)C_p + q$, and C_p is the constant in BDG inequality.

Then for any $(\xi, J) \in L^p \times \mathcal{S}^p$, BSDE (9) has a unique solution $(Y, Z \circ M) \in \mathcal{S}^p \times \mathcal{H}^p$ such that

$$\|Y\|_{\mathcal{S}^p} + \|Z \circ M\|_{\mathcal{H}^p} \leq K_p \left(\|\xi\|_{L^p} + \|J\|_{\mathcal{S}^p}\right),$$

where K_p is a positive constant independent of (ξ, J).

Proof. The proof is very analogous with Theorem 3.1. Now let us sketch the proof. For any $(y, z \circ M) \in \mathcal{S}^p \times \mathcal{H}^p$, consider the following BSDE

$$\begin{aligned} Y_t &= \xi + J_T - J_t + \int_{t+}^T f(s, y_{s-})\, d[N_1, N_2]_s \\ &\quad + \int_{t+}^T g(s, y_{s-}, z_s)\, d[N_3, M]_s - \int_{t+}^T Z_s\, dM_s. \end{aligned} \tag{10}$$

We define F as (3). According to Doob's inequality, we have

$$\begin{aligned} \|F\|_{\mathcal{S}^p} &\leq q\left\|\xi + J_T\right\|_{L^p} + q\left\|\int_0^T f(s, y_{s-})\, d[N_1, N_2]_s\right\|_{L^p} \\ &\quad + q\left\|\int_0^T g(s, y_{s-}, z_s)\, d[N_3, M]_s\right\|_{L^p}. \end{aligned}$$

In view of Lemmas 2.8 and 2.9, we have

$$\left\|\int_0^T f(s, y_{s-})\, d[N_1, N_2]_s\right\|_{L^p} \leq 2p\|y\|_{\mathcal{S}^p}\|N_2\|_{BMO}\|\alpha \circ N_1\|_{BMO},$$

and

$$\begin{aligned} \left\|\int_0^T g(s, y_{s-}, z_s)\, d[N_3, M]_s\right\|_{L^p} &\leq 2p\|y\|_{\mathcal{S}^p}\|\beta^{\frac{1}{2}} \circ M\|_{BMO}\|\beta^{\frac{1}{2}} \circ N_3\|_{BMO} \\ &\quad + \sqrt{2}p\|z \circ M\|_{\mathcal{H}^p}\|\gamma \circ N_3\|_{BMO}, \end{aligned}$$

which leads to that $F \in \mathcal{S}^p$. Since $\mathbb{P} \in \Gamma_e(M)$, by Lemma 2.7, we know that M has the strong property of predicable representation. Thus there exists a predictable process Z such that

$$F_t = F_0 + \int_0^t Z_s \, dM_s.$$

This implies

$$\begin{aligned} F_t + \int_{t+}^T Z_s dM_s &= \xi + J_T + \int_{t+}^T f(s, y_{s-}) d[N_1, N_2]_s \\ &\quad + \int_{t+}^T g(s, y_{s-}, z_s) d[N_3, M]_s. \end{aligned}$$

We define

$$Y_t := F_t - J_t - \int_0^t f(s, y_{s-}) \, d[N_1, N_2]_s - \int_0^t g(s, y_{s-}, z_s) \, d[N_3, M]_s.$$

It follows that $(Y, Z \circ M)$ is a solution of BSDE (10).

Note that

$$\begin{aligned} Y_t &= -J_t + \mathbb{E}\left[\xi + J_T | \mathscr{F}_t\right] + \mathbb{E}\left[\int_{t+}^T f(s, y_{s-}) \, d[N_1, N_2]_s \,\middle|\, \mathscr{F}_t\right] \\ &\quad + \mathbb{E}\left[\int_{t+}^T g(s, y_{s-}, z_s) \, d[N_3, M]_s \,\middle|\, \mathscr{F}_t\right]. \end{aligned}$$

In view of Doob's inequality, we have

$$\begin{aligned} \|Y\|_{\mathcal{S}^p} &\leq \|J\|_{\mathcal{S}^p} + q\|\xi + J_T\|_{L^p} \\ &\quad + 2pq\|\alpha \circ N_1\|_{BMO}\|N_2\|_{BMO}\|y\|_{\mathcal{S}^p} \\ &\quad + 2pq\|\beta^{\frac{1}{2}} \circ M\|_{BMO}\|\beta^{\frac{1}{2}} \circ N_3\|_{BMO}\|y\|_{\mathcal{S}^p} \\ &\quad + \sqrt{2}pq\|\gamma \circ N_3\|_{BMO}\|z \circ M\|_{\mathcal{H}^p}. \end{aligned}$$

According to BDG inequality, we have

$$\begin{aligned} \|Z \circ M\|_{\mathcal{H}^p} &\leq C_p \, \|Z \circ M\|_{\mathcal{S}^p} \leq 2C_p \left\| \int_{t^+}^T Z \, dM_s \right\|_{\mathcal{S}^p} \\ &\leq 2C_p\|Y\|_{\mathcal{S}^p} + 2C_p\|\xi + J_T\|_{L^p} + 2C_p\|J\|_{\mathcal{S}^p} \\ &\quad + 4pC_p\|y\|_{\mathcal{S}^p}\|N_2\|_{BMO}\|\alpha \circ N_1\|_{BMO} \end{aligned}$$

$$4pC_p\|y\|_{\mathcal{S}^p}\|\beta^{\frac{1}{2}}\circ M\|_{BMO}\|\beta^{\frac{1}{2}}\circ N_3\|_{BMO}$$
$$+2\sqrt{2}pC_p\|z\circ M\|_{\mathcal{H}^p}\|\gamma\circ N_3\|_{BMO}.$$

Concluding the above, we have

$$\begin{aligned}\|Y\|_{\mathcal{S}^p}+\|Z\circ M\|_{\mathcal{H}^p} \le\ & \overline{C}_p\|\xi+J_T\|_{L^p}+(1+4C_p)\,\|J\|_{\mathcal{S}^p} \\ & +2p\overline{C}_p\,\|\alpha\circ N_1\|_{BMO}\,\|N_2\|_{BMO}\,\|y\|_{\mathcal{S}^p} \\ & +2p\overline{C}_p\left\|\beta^{\frac{1}{2}}\circ M\right\|_{BMO}\left\|\beta^{\frac{1}{2}}\circ N_3\right\|_{BMO}\|y\|_{\mathcal{S}^p} \\ & +\sqrt{2}p\overline{C}_p\,\|\gamma\circ N_3\|_{BMO}\,\|z\circ M\|_{\mathcal{H}^p}\,. \end{aligned} \tag{11}$$

Then we get that the solution $(Y, Z\circ M)$ of BSDE (10) is in $\mathcal{S}^p\times\mathcal{H}^p$. The uniqueness can be easily proved if one estimates $\|Y^1-Y^2\|_{\mathcal{S}^p}+\|(Z^1-Z^2)\circ M\|_{\mathcal{H}^p}$ via a similar calculation to the one that led to (11).

We shall still use the contraction mapping principle to prove the existence and uniqueness of the solution. Similar to the proof of Theorem 3.1, since the martingale $(N_2, \beta^{\frac{1}{2}}\circ M, \gamma\circ N_3, \beta^{\frac{1}{2}}\circ N_3)$ is $\vec{\varepsilon}$-sliceable in $(BMO)^4$ with the corresponding finite sequence of stopping times $\{T_i,\ \ i=0,\dots,\widetilde{I}+1\}$, we can show that the BSDE

$$\begin{aligned} Y_t\ =\ & Y^{i+1}_{T_{i+1}}+(J_{T_{i+1}}-J_t)+\int_{t+}^{T_{i+1}} f(s,Y_{s-})\,d[N_{1i},N_{2i}]_s \\ & +\int_{t+}^{T_{i+1}} g(s,Y_{s-},Z_s)\,d[N_{3i},M_i]_s-\int_{t+}^{T_{i+1}} Z_s\,dM_{is}, \end{aligned}$$

has a unique solution $(Y^i, Z^i\circ M_i)$ in $\mathcal{S}^p_i\times\mathcal{H}^p_i$ for $i=0,1,\cdots,\widetilde{I}$, inductively in a backward way. Then, the process $(Y, Z\circ M)$ given by

$$Y_t:=\sum_{i=0}^{\widetilde{I}} Y^i_t\chi_{[T_i,T_{i+1})}(t)\quad\text{and}\quad Z_t:=\sum_{i=0}^{\widetilde{I}} Z^i_t\chi_{[T_i,T_{i+1})}(t)$$

lies in $\mathcal{S}^p\times\mathcal{H}^p$ and is the unique adapted solution to BSDE (9). Moreover, we have

$$\begin{aligned} & (1-\rho_p)\left(\|Y\|_{\mathcal{S}^p}+\left\|Z^i\circ M\right\|_{\mathcal{H}^p}\right) \\ \le\ & \overline{C}_p\,\|\xi+J_T\|_{L^p}+(4C_p+1)\,\|J\|_{\mathcal{S}^p}\,. \end{aligned} \tag{12}$$

This completes the proof. ∎

4. Linear BSDEs with jumps

In this section, we consider BSDE (1) with $f = 0, J = 0$ and g being linear with z and independent of y:

$$(13)\quad Y_t = \xi + \int_{t+}^{T} \gamma(s) Z_s \, d[N_3, M]_s - \int_{t+}^{T} Z_s \, dM_s - \int_{t+}^{T} dM_s^{\perp}, \quad t \in [0, T].$$

In Theorem 3.2, the terminal value ξ is assumed to be essentially bounded to get that $Z \circ M \in BMO$. For the special BSDE (13), it is sufficient to assume that $\xi \in BMO$ so as to guarantee that $Z \circ M \in BMO$.

Theorem 4.1. *Assume that $\gamma \circ N_3$ is ε-sliceable in BMO such that*

$$\varepsilon < (4\sqrt{5} + \sqrt{10} c_1^{-1})^{-1},$$

where c_1 is the constant in the BDG inequality for $p = 1$. Then for any $\xi \in BMO$, BSDE (13) has a unique solution $(Y, Z \circ M, M^{\perp}) \in \bigcap_{p>1} \mathcal{S}^p \times (BMO)^2$ such that

$$\|Z \circ M\|_{BMO} + \left\|M^{\perp}\right\|_{BMO} \le K \|\xi\|_{BMO}$$

where K is a positive constant independent of ξ.

To prove Theorem 4.1, firstly we need to prove the following lemma.

Lemma 4.1. *If $X, M \in BMO$, we have*

$$\|[X, M]_T\|_{BMO} \le (4\sqrt{5} + \sqrt{10} c_1^{-1}) \|X\|_{BMO} \|M\|_{BMO},$$

where c_1 is the constant in BDG inequality for $p = 1$.

Proof of Lemma 4.1. Take $Y \in \mathcal{H}^1$. Using stopping times $\{\tau_n, n = 1, 2, \ldots\}$ to make all the processes mentioned in this lemma bounded for any fixed n, we have

$$\begin{aligned}
& \left|\mathbb{E}\left[[Y, \mathbb{E}^{\mathscr{F}_{\cdot}}[X, M]_T]_T^{\tau_n}\right]\right| \\
= & \left|\mathbb{E}\left[Y_T^{\tau_n} \mathbb{E}^{\mathscr{F}_T}[X, M]_T\right]\right| = \left|\mathbb{E}\left[Y_T^{\tau_n}[X, M]_T\right]\right| \\
= & \left|\mathbb{E}\left[\int_0^T Y_{s-}^{\tau_n} d[X, M]_s + \int_0^T [X, M]_{s-} dY_s^{\tau_n} + [Y^{\tau_n}, [X, M]]_T\right]\right| \\
= & \left|\mathbb{E}\left[[Y_-^{\tau_n} \circ X, M]_T + [Y^{\tau_n}, [X, M]]_T\right]\right|
\end{aligned}$$

$$\begin{aligned} &= \left|\mathbb{E}\left[[Y_-^{\tau_n}\circ X, M]_T\right] + \mathbb{E}\left[\sum_{0<s\leq T}\Delta Y_s^{\tau_n}\Delta[X,M]_s\right]\right| \\ &\leq \left|\mathbb{E}\left[[Y_-^{\tau_n}\circ X, M]_T\right]\right| + \mathbb{E}\left[\sup_{0<s\leq T}|\Delta X_s|\sum_{0<s\leq T}|\Delta Y_s^{\tau_n}\Delta M_s|\right], \end{aligned}$$

where $\mathbb{E}^{\mathscr{F}_t}[\cdot] := \mathbb{E}[\cdot|\mathscr{F}_t]$, $0 \leq t \leq T$. According to the discontinuous BMO martingale theory (see e.g. [19, page 200]), if $X \in BMO$, then X has bounded jumps satisfying

$$\sup_{0<s\leq T}|\Delta X_s| \leq 2\sqrt{2}\|X\|_{BMO}.$$

Thus we have

$$\begin{aligned} \left|\mathbb{E}\left[[Y, \mathbb{E}^{\mathscr{F}_\cdot}[X,M]_T]_T^{\tau_n}\right]\right| &\leq \left|\mathbb{E}\left[[Y_-^{\tau_n}\circ X, M]_T\right]\right| \\ &\quad +2\sqrt{2}\|X\|_{BMO}\left|\mathbb{E}\left[\int_0^T|d[Y^{\tau_n},M]_s|\right]\right|. \end{aligned}$$

According to Fefferman's inequality and Lemma 2.9, we have

$$\begin{aligned} \left|\mathbb{E}\left[[Y, \mathbb{E}^{\mathscr{F}_\cdot}[X,M]_T]_T^{\tau_n}\right]\right| &\leq \sqrt{2}\|Y_-^{\tau_n}\circ X\|_{\mathcal{H}^1}\|M\|_{BMO} \\ &\quad +4\|X\|_{BMO}\|Y^{\tau_n}\|_{\mathcal{H}^1}\|M\|_{BMO} \\ &\leq \sqrt{2}\|Y^{\tau_n}\|_{\mathcal{S}^1}\|X\|_{BMO}\|M\|_{BMO} \\ &\quad +4\|X\|_{BMO}\|Y^{\tau_n}\|_{\mathcal{H}^1}\|M\|_{BMO}. \end{aligned}$$

Using Lemma 2.8 and the BDG inequality, it follows

$$\begin{aligned} \left|\mathbb{E}\left[[Y, \mathbb{E}^{\mathscr{F}_\cdot}[X,M]_T]_T^{\tau_n}\right]\right| &\leq \sqrt{2}c_1^{-1}\|Y^{\tau_n}\|_{\mathcal{H}^1}\|X\|_{BMO}\|M\|_{BMO} \\ &\quad +4\|X\|_{BMO}\|Y^{\tau_n}\|_{\mathcal{H}^1}\|M\|_{BMO} \\ &\leq \sqrt{2}c_1^{-1}\|Y\|_{\mathcal{H}^1}\|X\|_{BMO}\|M\|_{BMO} \\ &\quad +4\|X\|_{BMO}\|Y\|_{\mathcal{H}^1}\|M\|_{BMO}. \end{aligned}$$

In view of the well-known Fatou's Lemma, we can get that

$$\left|\mathbb{E}\left[[Y, \mathbb{E}^{\mathscr{F}_\cdot}[X,M]]_T\right]\right| \leq (4+\sqrt{2}c_1^{-1})\|X\|_{BMO}\|Y\|_{\mathcal{H}^1}\|M\|_{BMO}.$$

Using Lemma 2.5, we have

$$\begin{aligned}\|[X,M]_T\|_{BMO} &= \|\mathbb{E}^{\mathscr{F}_\cdot}[X,M]_T\|_{BMO}\\ &\le \sqrt{5}\sup_{\|Y\|_{\mathcal{H}^1}\le 1}\left\{\left|\mathbb{E}\left[[Y,E^{\mathscr{F}_\cdot}[X,M]]_T\right]\right|\right\}\\ &\le (4\sqrt{5}+\sqrt{10}c_1^{-1})\|X\|_{BMO}\|M\|_{BMO}.\end{aligned}$$

This completes the proof. □

Proof of Theorem 4.1. For any $z\circ M\in BMO$, consider the following BSDEs:

$$Y_t=\xi+\int_{t+}^T\gamma(s)z_s\,d[N_3,M]_s-\int_{t+}^T Z_s\,dM_s-\int_{t+}^T dM_s^{\perp}. \tag{14}$$

We define

$$F_t:=\mathbb{E}\left[\xi+\int_0^T\gamma(s)z_s\,d[N_3,M]_s\,\middle|\,\mathscr{F}_t\right].$$

By Doob's inequality and Lemma 2.9, we know that $F\in\mathcal{S}^p$ for any $p\in(1,\infty)$. According to Lemma 2.6, we know that BSDE (14) admits a solution $(Y,Z\circ M,M^{\perp})$ such that

$$Y_t=\mathbb{E}\left[\xi+\int_{t+}^T\gamma(s)z_s\,d[N_3,M]_s\,\middle|\,\mathscr{F}_t\right]\in\mathcal{S}^p$$

for any $p\in(1,\infty)$, and

$$(Z\circ M+M^{\perp})_t=\mathbb{E}\left[\xi|\mathscr{F}_t\right]+\mathbb{E}\left[\int_0^T\gamma(s)z_s\,d[N_3,M]_s\,\middle|\,\mathscr{F}_t\right]-Y_0.$$

According to Lemma 4.1, we have

$$\begin{aligned}\|Z\circ M\|_{BMO} &\le \|Z\circ M+M^{\perp}\|_{BMO}\\ &\le \|\xi\|_{BMO}+\|[\gamma\circ N_3,z\circ M]_T\|_{BMO}\\ &\le \|\xi\|_{BMO}+(4\sqrt{5}+\sqrt{10}c_1^{-1})\|\gamma\circ N_3\|_{BMO}\|z\circ M\|_{BMO}.\end{aligned}$$

We are going to use the contraction mapping principle to prove the existence and uniqueness of the solution. Consider the following map I in the Banach space BMO: for $z\circ M\in BMO$, define $I(z\circ M)$ to be the component $Z\circ M$ of the unique adapted solution $(Y,Z\circ M,M^{\perp})$ of the

BSDE (14). Let $z^i \circ M \in BMO$ with $i = 1, 2$. Denote by $Z^i \circ M$ the image $I(z^i \circ M)$ for $i = 1, 2$. Similar to the above arguments, we can show that

$$\begin{aligned}\left\|(Z^1 - Z^2) \circ M\right\|_{BMO} &\leq (4\sqrt{5} + \sqrt{10}c_1^{-1}) \left\|\gamma \circ N_3\right\|_{BMO} \\ &\quad \times \left\|(z^1 - z^2) \circ M\right\|_{BMO}.\end{aligned}$$

The rest of the proof is identical to that of Theorem 3.1. This completes the proof. □

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