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Approximation of BSDEs with super-linearly growing generators by Euler's polygonal line method: A simple proof of the existence

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ABSTRACT

This paper develops the Euler's polygonal line method for the backward stochastic differential equations (BSDEs) with super-linearly growing generators. The generators are allowed to be super-linearly growing in the first unknown variable y and sub-quadratic growing in the second unknown variable z when the monotonicity condition is satisfied. The convergence rate of the Euler approximation is derived, which also provides a simple proof for the existence of the solution to the monotone BSDEs. The proof is very simple and short, without involving the conventional techniques of truncating and smoothing on the generators.

1. Introduction

In this paper we consider the Euler's polygonal line method for the following nonlinear backward stochastic differential equations (BSDEs):

$$\begin{cases} dY_t &= -f(t, Y_t, Z_t)\,dt + Z_t\,dW_t, \qquad t \in [0,T), \\ Y_T &= \xi, \end{cases} \tag{1.1}$$

where the generator $f : \Omega \times \mathbb{R}^+ \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longrightarrow \mathbb{R}^k$ satisfies the monotonicity condition in the first unknown variable y and sub-quadratic growth in the second unknown variable z. Nonlinear BSDEs are initially introduced by Pardoux and Peng [1], who establish an existence and uniqueness result of solutions under the uniformly Lipschitz assumption. Since then, much effort has been given to weaken their classical assumption. The interest of extensions of the classical results is inspired by the fact that, in many applications, the usual Lipschitz condition fails to be satisfied.

The monotonicity condition for BSDEs appears for the first time in a paper by Peng [2], where the generators satisfy the monotonicity condition in y, which is very helpful to study the BSDEs with unbounded stochastic terminal times and super-linearly growing generators. Pardoux [3] and Pardoux and Rascanu [4] solve this kind of BSDEs in the space $\mathcal{S}_T^2(\mathbb{R}^k) \times \mathcal{H}_T^2(\mathbb{R}^{k\times d})$ by using several operations on the generators, including truncation, smoothing, passing to the limit and localization. These techniques are also extended to allow the monotonicity coefficients to be stochastic processes, and the BSDEs are solved in the space $\mathcal{S}_T^p(\mathbb{R}^k) \times \mathcal{H}_T^p(\mathbb{R}^{k\times d})$ with $p \neq 2$.

Systems of quadratic BSDEs, whose generators have a quadratic growth with respect to the variable z, arise naturally in stochastic optimal control and stochastic differential games, such as linear quadratic optimal control with random coefficients and risk-sensitive optimal control. For the scalar case, Kobylanski [5] gives the existence result under the assumption that the generators quasi-linearly grow in y. The uniqueness result is derived under the assumption that the derivatives $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ have some bounded properties. Briand and Hu [6] extend these results. For the existence of the solution, they use the truncating technique to allow the generators to be monotone in y and the terminal value to be unbounded. And they require the generators be convex or concave with respect to the variable z to get the uniqueness of the solution. Recently, Hu and Tang [7] solve the *multi*-dimensional quadratic BSDEs under the diagonally quadratic condition by using the *BMO* martingale theory.

For most nonlinear BSDEs, explicit solutions are not available in general. It is theoretically and practically appealing to develop approximating methods for BSDEs. There have been many algorithms for computing solutions of BSDEs. A four step scheme is developed by Ma, Protter and Yong [8] to approximate forward backward stochastic partial differential equations (FBSDEs) under some regularity assumptions. Gobet, Lemor and Warin [9] and Zhang [10] consider the decoupled FBSDEs with the assumption

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of uniformly Lipschitz continuity in (y, z). Under this assumption, Peng and Xu [11] develop a simple Euler scheme to solve the one-dimensional BSDEs in a time-discrete way. Richou [12] and Chassagneux and Richou [13] investigate the numerical approximation of Markovian BSDEs with generators of quadratic growth in z and uniform Lipschitz continuity in y. By studying the corresponding BSΔEs, Cheridito and Stadje [14] propose an approximating scheme for scalar BSDEs with sub-quadratic growth in z, and they give the convergence results without convergence rate. All these results depend strongly on the uniformly continuous assumption in y. The new features of this paper are that the generators f are allowed to be non-uniformly Lipschitz continuous in y and the convergence rates are given.

Lionnet, Reis and Szpruch [15] consider the Markovian FBSDEs with polynomial growth in y, and prove that some implicit θ-type schemes and a tamed version of the explicit Euler scheme are convergent for their FBSDEs. Subsequently, they [16] propose some modified explicit schemes, which have the same rate of convergence as standard implicit schemes and the similar computational cost to the standard explicit scheme.

In this article, we establish the Euler's polygonal line method to approximate our non-Markovian BSDEs with monotone generators, which is different from the schemes in [15,16] designed for the Markovian FBSDEs. Krylov [17] first used the Euler's polygonal line method to prove the solvability for monotone stochastic differential equations (SDEs), and the proof is very simple and short among the relevant literature. We shall show that this method can be developed to solve monotone BSDEs in some proper spaces, which seems to be new. We first consider the scalar BSDEs with the generators being non-uniformly Lipschitz continuous in the first unknown variable y and sub-quadratic in the second unknown variable z (i.e., the growth in z is dominated by $1+|z|^{2-\varepsilon}$ for a positive number $\varepsilon \in (0, 2]$). The proper space to prove the convergence results is $\mathcal{S}_T^{\infty}(\mathbb{R}) \times BMO$ and the convergence speed is at least $\frac{\varepsilon}{4}$. We also consider the multi-dimensional BSDEs with the generators being super-linearly growing in the first unknown variable y and uniformly Lipschitz continuous in the second unknown variable z. Motivated by the explicit Euler scheme for SDEs in [18], we show that the approximating solution (Y^n, Z^n) converges to the exact solution (Y, Z) in $\mathcal{S}_T^p(\mathbb{R}^k) \times \mathcal{H}_T^p(\mathbb{R}^{k\times d})$ with $p \geq 1$ and the rate of convergence is $\frac{1}{2}$.

The rest of the paper is organized as follows. In Section 2, we prepare some basic notations. In Section 3, we state the Euler's polygonal line methods and their convergence theorems. The proofs are given in Section 4.

2. Preliminaries

The norm of a $k \times d$ matrix is given by $|y| = \sqrt{\langle y, y\rangle}$, where $\langle y_1, y_2\rangle := trace(y_1 y_2^*)$. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space on which a d'-dimensional Wiener process $W = \{W_t\,;\ t \geq 0\}$ is defined such that $\{\mathscr{F}_t\}_{t\geq 0}$ is the natural filtration generated by W, augmented by all the $\mathbb{P}$-null sets in $\mathscr{F}$. The terminal time T is a fixed positive real number. For $0 \leq T_1 \leq T_2 \leq T$ and $p \geq 1$,

(i) $\mathcal{L}_{T_1}^p(\mathbb{R}^k)$ is the space of all $\mathscr{F}_{T_1}$-measurable random variables $X : \Omega \longrightarrow \mathbb{R}^k$ such that

$$\|X\|_{\mathcal{L}_{T_1}^p(\mathbb{R}^d)} = \left\{\mathbb{E}\left[|X|^p\right]\right\}^{\frac{1}{p}} < \infty;$$

and $\mathcal{L}_{T_1}^{\infty}(\mathbb{R}^k)$ is the space of all essentially bounded $\mathscr{F}_{T_1}$-measurable random variables;

(ii) $\mathcal{H}_{[T_1,T_2]}^p(\mathbb{R}^{k\times d})$ is the space of all predictable processes $\phi : \Omega \times [T_1, T_2] \longrightarrow \mathbb{R}^{k\times d}$ such that

$$\|\phi\|_{\mathcal{H}_{[T_1,T_2]}^p(\mathbb{R}^{k\times d})} = \left\{\mathbb{E}\left[\left(\int_{T_1}^{T_2} |\phi_t|^2\,dt\right)^{\frac{p}{2}}\right]\right\}^{\frac{1}{p}} < \infty;$$

(iii) $\mathcal{S}_{[T_1,T_2]}^p(\mathbb{R}^k)$ is the space of all adapted càdlàg processes $\phi : \Omega \times [T_1, T_2] \longrightarrow \mathbb{R}^k$ such that

$$\|\phi\|_{\mathcal{S}_{[T_1,T_2]}^p(\mathbb{R}^k)} = \left\{\mathbb{E}\left[\sup_{T_1\leq t\leq T_2} |\phi_t|^p\right]\right\}^{\frac{1}{p}} < \infty;$$

and $\mathcal{S}_{[T_1,T_2]}^{\infty}(\mathbb{R}^k)$ is the space of all adapted càdlàg processes ϕ such that

$$\|\phi\|_{\mathcal{S}_{[T_1,T_2]}^{\infty}(\mathbb{R}^k)} = \left\|\sup_{T_1\leq t\leq T_2} |\phi_t|\right\|_{\mathcal{L}_{T_2}^{\infty}(\mathbb{R}^k)} < \infty;$$

(iv) BMO is the space of all uniformly integrable càdlàg martingales M with $M_0 = 0$ such that

$$\|M\|_{BMO} := \left(\sup_{\tau} \|\mathbb{E}\left[\langle M\rangle_T - \langle M\rangle_\tau \,|\mathscr{F}_\tau\right]\|_{\mathcal{L}_T^{\infty}(\mathbb{R})}\right)^{\frac{1}{2}} < \infty,$$

where the supremum is taken over all stopping times $\tau \leq T$.

For simplicity, we set $\mathcal{H}_T^p(\mathbb{R}^{k\times d}) := \mathcal{H}_{[0,T]}^p(\mathbb{R}^{k\times d})$ and $\mathcal{S}_T^p(\mathbb{R}^k) := \mathcal{S}_{[0,T]}^p(\mathbb{R}^k)$. For any fixed $p \in [1, \infty]$, $q \in [1, \infty)$, $n \in \mathbb{N}_+$, $t \in [0, T)$ and $i = 1, 2, \ldots, n$, define

$$T_i := \frac{i}{n}T, \qquad \varphi(n, t) := \frac{\left[\frac{nt}{T}\right] + 1}{n}T, \qquad \varphi(n, T) := T,$$

and

$$\mathcal{S}^{p,i}(\mathbb{R}^k) := \mathcal{S}_{[T_{n-i},T_{n-i+1}]}^p(\mathbb{R}^k), \qquad \mathcal{H}^{q,i}(\mathbb{R}^{k\times d}) := \mathcal{H}_{[T_{n-i},T_{n-i+1}]}^q(\mathbb{R}^{k\times d}).$$

Throughout the paper, by saying that a vector-valued or matrix-valued function belongs to a function space, we mean all the components belong to that space. By $C > 0$, we denote a generic constant, which in particular does not depend on the number of meshes n and possibly changes from line to line. For simplicity of notations, we omit the argument ω of stochastic processes if there is no danger of confusion.

3. Euler approximation and the convergence results

3.1. Sub-quadratic growth in z

In this section we state our main results. We first consider the scalar BSDEs with generators being sub-quadratic growing in z and non-uniformly Lipschitz continuous in y, such as $f(t, y, z) := -sign(y)\sqrt{|y|} + |z|^{\frac{3}{2}}$. Let $k = 1$. We make the following assumptions:

(Z1) (Monotonicity) There exists a constant $K > 0$ such that

$$2\left\langle y_1 - y_2, f(t, y_1, z) - f(t, y_2, z)\right\rangle \leq K\,|y_1 - y_2|^2\,, \tag{3.1}$$

for any $(\omega, t, y_1, y_2, z) \in \Omega \times [0, T] \times (\mathbb{R})^2 \times \mathbb{R}^d$;

(Z2) (Continuity in z) There exist a constant $F > 0$ and a non-decreasing function $\rho : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that

$$|f(t, y, z_1) - f(t, y, z_2)| \leq F\left(1 + \rho\left(|y|\right) + |z_1| + |z_2|\right)|z_1 - z_2|\,, \tag{3.2}$$

for any $(\omega, t, y, z_1, z_2) \in \Omega \times [0, T] \times \mathbb{R} \times \left(\mathbb{R}^d\right)^2$;

(Z3) (Growth condition) There exist four constants $\alpha > 0$, $\beta \geq 0$, $\gamma > 0$ and $\varepsilon \in (0, 2]$ such that

$$|f(t, y, z)| \leq \alpha + \beta\,|y| + \frac{\gamma}{2}\,|z|^{2-\varepsilon}\,, \tag{3.3}$$

for any $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$;

(Z4) The map $y \mapsto f(t, y, z)$ is continuous from $\mathbb{R}$ to $\mathbb{R}$ for each $(\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d$;

(**Z5**) The terminal value ξ is in $\mathcal{L}_T^\infty(\mathbb{R})$.

Since the BSDEs satisfy the preceding assumptions, the solvability is obtained from [6, Corollary 4] and [19, Theorem 2.1] as follows.

Lemma 3.1. *Let Assumptions (**Z1**)–(**Z5**) be satisfied. Then BSDE (1.1) admits a unique solution $(Y, Z\circ W)$ in $\mathcal{S}_T^\infty(\mathbb{R})\times\mathcal{H}_T^2(\mathbb{R}^d)$.*

The classical proof of the above result involves several technical methods, including truncating, smoothing, passing to a limit on the generators and localization technique. In the next section, we will use the Euler approximation to prove the above result, which is very simple and seems to be novel.

Before giving the approximation equations, let us first consider a simple class of BSDEs with the generators $\bar f$ being independent with y, i.e.,

$$\bar f = \bar f(t,z) : \Omega\times[0,T]\times\mathbb{R}^d\to\mathbb{R}.$$

The solvability of such BSDEs can be found in Hu and Tang [7, Lemma 2.1] as follows, which is the basic tool to design our approximating scheme.

Lemma 3.2. *Assume that $\xi\in\mathcal{L}_T^\infty(\mathbb{R})$ and there exist two positive constants $\bar\alpha,\bar\gamma$ such that*

$$\left|\bar f(t,z)\right| \le \bar\alpha + \frac{\bar\gamma}{2}|z|^2, \qquad |f(s,z_1)-f(s,z_2)|\le\bar\alpha\,(1+|z_1|+|z_2|)\,|z_1-z_2|,$$

for all $(\omega,t,z,z_1,z_2)\in\Omega\times[0,T]\times(\mathbb{R}^d)^3$. Then the BSDE $(\bar f, T,\xi)$ has a solution $(Y, Z\circ W)$ in $\mathcal{S}_T^\infty(\mathbb{R})\times\mathcal{H}_T^2(\mathbb{R}^d)$ such that

$$\|Y\|_{\mathcal{S}_T^\infty(\mathbb{R})}\le\bar\alpha T+\|\xi\|_{\mathcal{S}_T^\infty(\mathbb{R})}.$$

Now we give the Euler's polygonal line method. Consider the following BSDE:

$$Y_t^{n,1}=\xi+\int_t^T f\left(s,\mathbb{E}^{\mathscr{F}_s}[\xi], Z_s^{n,1}\right)ds-\int_t^T Z_s^{n,1}\,dW_s,\qquad t\in[T_{n-1},T]. \tag{3.4}$$

The generator of BSDE (3.4) is

$$f^{n,1}(s,z):=f\left(s,\mathbb{E}^{\mathscr{F}_s}[\xi],z\right),$$

which is independent with y. Note that $|z|^{2-\varepsilon}\le 1+|z|^2$, we have

$$\left|f^{n,1}(s,z)\right|\le\alpha+\beta\,\|\xi\|_{\mathcal{L}_T^\infty(\mathbb{R})}+\frac{\gamma}{2}|z|^{2-\varepsilon}\le\alpha+\frac{\gamma}{2}+\beta\,\|\xi\|_{\mathcal{L}_T^\infty(\mathbb{R})}+\frac{\gamma}{2}|z|^2.$$

One can verify that $f^{n,1}$ satisfies all the conditions in Lemma 3.2 with the coefficients:

$$\bar\alpha^{n,1}=\alpha+\frac{\gamma}{2}+\beta\,\|\xi\|_{\mathcal{L}_T^\infty(\mathbb{R})},\quad \bar\gamma^{n,1}=\gamma.$$

By Lemma 3.2, BSDE (3.4) has a solution $(Y^{n,1},Z^{n,1})$ in $\mathcal{S}^{\infty,1}(\mathbb{R})\times\mathcal{H}^{2,1}(\mathbb{R}^d)$ such that

$$\|Y^{n,1}\|_{\mathcal{S}^{\infty,1}(\mathbb{R})}\le\left(\alpha+\frac{\gamma}{2}\right)\frac{T}{n}+\left(1+\frac{\beta T}{n}\right)\|\xi\|_{\mathcal{L}_T^\infty(\mathbb{R})}.$$

Next we consider the following BSDE:

$$Y_t^{n,2}=Y_{T_{n-1}}^{n,1}+\int_t^{T_{n-1}} f\left(s,\mathbb{E}^{\mathscr{F}_s}\left[Y_{T_{n-1}}^{n,1}\right],Z_s^{n,1}\right)ds-\int_t^{T_{n-1}}Z_s^{n,1}\,dW_s,$$
$$t\in[T_{n-2},T_{n-1}]. \tag{3.5}$$

The generator of BSDE (3.5) is

$$f^{n,2}(s,y,z):=f\left(s,\mathbb{E}^{\mathscr{F}_s}\left[Y_{T_{n-1}}^{n,1}\right],z\right),$$

which does not depend on y. Note that

$$\left|f^{n,2}(s,y,z)\right|\le\alpha+\frac{\gamma}{2}+\beta\left\|Y^{n,1}\right\|_{\mathcal{S}^{\infty,1}(\mathbb{R})}+\frac{\gamma}{2}|z|^2.$$

Thus $f^{n,2}$ satisfies all the conditions in Lemma 3.2 with

$$\bar\alpha^{n,2}=\alpha+\frac{\gamma}{2}+\beta\left\|Y^{n,1}\right\|_{\mathcal{S}^{\infty,1}(\mathbb{R})},\quad\bar\gamma^{n,2}=\gamma.$$

By Lemma 3.2, BSDE (3.5) has a solution $(Y^{n,2},Z^{n,2})\in\mathcal{S}^{\infty,2}(\mathbb{R})\times\mathcal{H}^{2,2}(\mathbb{R}^d)$ such that

$$\|Y^{n,2}\|_{\mathcal{S}^{\infty,2}(\mathbb{R})}\le\left(\alpha+\frac{\gamma}{2}\right)\frac{T}{n}+\left(1+\frac{\beta T}{n}\right)\left\|Y^{n,1}\right\|_{\mathcal{S}^{\infty,1}(\mathbb{R})}.$$

Inductively in a backward way, one can use Lemma 3.2 to show that for each $i=1,\ldots,n$, the following BSDE:

$$Y_t^{n,i}=Y_{T_{n-i+1}}^{n,i-1}+\int_t^{T_{n-i+1}} f\left(s,\mathbb{E}^{\mathscr{F}_s}\left[Y_{T_{n-i+1}}^{n,i-1}\right],Z_s^{n,i}\right)ds$$
$$-\int_t^{T_{n-i+1}}Z_s^{n,i}\,dW_s,\quad t\in[T_{n-i},T_{n-i+1}],$$

has a solution $(Y^{n,i},Z^{n,i})$ in $\mathcal{S}^{\infty,i}(\mathbb{R})\times\mathcal{H}^{2,i}(\mathbb{R}^d)$, where $Y_T^{n,0}=\xi$. The generator is

$$f^{n,i}(s,y,z):=f\left(s,\mathbb{E}^{\mathscr{F}_s}\left[Y_{T_{n-i+1}}^{n,i}\right],z\right)$$

with the parameters

$$\bar\alpha^{n,i}=\alpha+\frac{\gamma}{2}+\beta\left\|Y^{n,i-1}\right\|_{\mathcal{S}^{\infty,i-1}(\mathbb{R})},\quad\bar\beta^{n,i}=0,\quad\bar\gamma^{n,i}=\gamma.$$

Thus

$$\left\|Y^{n,i}\right\|_{\mathcal{S}^{\infty,i}(\mathbb{R})}\le\left(\alpha+\frac{\gamma}{2}\right)\frac{T}{n}+\left(1+\frac{\beta T}{n}\right)\left\|Y^{n,i-1}\right\|_{\mathcal{S}^{\infty,i-1}(\mathbb{R})}. \tag{3.6}$$

We define for $t\in[0,T]$,

$$Y_t^n:=\sum_{i=1}^n Y_t^{n,i}\cdot\chi_{[T_{n-i},T_{n-i+1})}+\xi\cdot\chi_{\{t=T\}}$$

and

$$Z_t^n:=\sum_{i=1}^n Z_t^{n,i}\cdot\chi_{[T_{n-i},T_{n-i+1})}+Z_T^{n,1}\cdot\chi_{\{t=T\}}.$$

Thus we know that (Y^n,Z^n) is in $\mathcal{S}_T^\infty(\mathbb{R})\times\mathcal{H}_T^2(\mathbb{R}^d)$ and is a solution of the following backward equation:

$$Y_t^n=Y_{\varphi(n,t)}^n+\int_t^{\varphi(n,t)} f\left(s,\mathbb{E}^{\mathscr{F}_s}\left[Y_{\varphi(n,s)}^n\right],Z_s^n\right)ds-\int_t^{\varphi(n,t)}Z_s^n\,dW_s,$$
$$t\in[0,T], \tag{3.7}$$

where $Y_T^n=\xi$.

We can prove that the approximated solution $\{(Y^n,Z^n)\}_{n\ge1}$ defined by (3.7) is a Cauchy sequence, which gives the existence result of solution and the error estimate as follows.

Theorem 3.1. *Let Assumptions (**Z1**)–(**Z5**) be satisfied. Then $\{(Y^n, Z^n)\}_{n\ge1}$ is a Cauchy sequence in $\mathcal{S}_T^\infty(\mathbb{R})\times BMO$, which converges to the unique solution $(Y, Z\circ W)$ of BSDE (1.1). Moreover, we have*

$$\left\|Y^n-Y\right\|_{\mathcal{S}_T^\infty(\mathbb{R})}+\left\|Z^n-Z\right\|_{BMO}\le\frac{C}{n^{\frac{\varepsilon}{4}}},$$

where C is a constant independent of n.

3.2. Super-linearly growth in y

Next we consider the Euler approximation for the BSDEs with generators being super-linearly growing in y and uniformly Lipschitz continuous in z, such as $f(t,y,z):=-y^m+\sin(z)$ with m being any positive odd integer. We make the following assumptions:

(Y1) (Monotonicity) There exist two constants $K > 0$ and $H > 0$ such that

$$2\langle y_1 - y_2, f(t, y_1, z) - f(t, y_2, z)\rangle \le K|y_1 - y_2|^2, \tag{3.8}$$

and

$$\langle y, f(t, y, 0)\rangle \le H\left(1 + |y|^2\right), \tag{3.9}$$

for any $(\omega, t, y, y_1, y_2, z) \in \Omega \times [0, T] \times \left(\mathbb{R}^k\right)^3 \times \mathbb{R}^{k\times d}$;

(Y2) (Continuity in z) There exists a constant $L > 0$ such that

$$|f(t, y, z_1) - f(t, y, z_2)| \le L|z_1 - z_2|, \tag{3.10}$$

for any $(\omega, t, y, z_1, z_2) \in \Omega \times [0, T] \times \mathbb{R}^k \times \left(\mathbb{R}^{k\times d}\right)^2$.

(Y3) (Continuity in y) There exist two constants $J > 0$ and $l > 0$ such that

$$|f(t, y_1, z) - f(t, y_2, z)| \le J\left(1 + |y_1|^l + |y_2|^l\right)|y_1 - y_2|, \tag{3.11}$$

for any $(\omega, t, y_1, y_2, z) \in \Omega \times [0, T] \times \left(\mathbb{R}^k\right)^2 \times \mathbb{R}^{k\times d}$.

(Y4) There exist a constant $M > 0$ such that

$$|f(t, 0, 0)| \le M, \tag{3.12}$$

for any $(\omega, t) \in \Omega \times [0, T]$.

(Y5) The terminal value $\xi \in \mathcal{L}_T^{p_0}\left(\mathbb{R}^k\right)$ for some $p_0 > 1$.

Under the above assumptions, we know there exists a positive constant C such that

$$|f(t, y, z)| \le C\left(1 + |y|^{l+1} + |z|\right). \tag{3.13}$$

For this kind of BSDEs, we have the following uniqueness and existence of solution from [20, Theorem 4.2].

Lemma 3.3. *Let the assumptions* (**Y1**)–(**Y5**) *be satisfied. Then BSDE (1.1) admits a unique solution* $(Y, Z) \in \mathcal{S}_T^{p_0}(\mathbb{R}^k) \times \mathcal{H}_T^{p_0}(\mathbb{R}^{k\times d})$.

Motivated by Sabanis [18], we shall give the approximation scheme when the generators are super-linearly growing in y. To do this, we introduce the approximate generators $\{f^n\}_{n\ge1}$ to calculate the approximate solutions $\{(Y^n, Z^n)\}_{n\ge1}$. For $n \in \mathbb{N}_+$ and $\alpha \in (0, 1)$, define

$$f^n(t, y, z) := f(t, y, z) - \frac{n^{-\alpha}|y|^l}{1 + n^{-\alpha}|y|^l} f(t, y, 0).$$

We observe that the value of α is closely related to the convergence rate of $\{f^n\}_{n\ge1}$. We shall leave α to be determined later in an optimal way. Note that

$$f^n(t, y, z) = f(t, y, z) - f(t, y, 0) + \frac{f(t, y, 0)}{1 + n^{-\alpha}|y|^l}.$$

Further, we observe that for any $(\omega, t, y, z, z_1, z_2) \in \Omega \times [0, T] \times \mathbb{R}^k \times (\mathbb{R}^{k\times d})^3$,

(i) for the constants C and L in (3.10) and (3.13), we have

$$\left|f^n(t, y, z)\right| \le Cn^\alpha(1 + |y|) + L|z|;$$

(ii) for the constant L in (3.10), we have

$$\left|f^n(t, y, z_1) - f^n(t, y, z_2)\right| \le L|z_1 - z_2|;$$

(iii) for the constant H in (3.9), we have

$$\left\langle y, f^n(t, y, 0)\right\rangle \le H\left(1 + |y|^2\right).$$

Now we give the Euler's polygonal line method. Consider the following BSDE:

$$Y_t^{n,1} = \xi + \int_t^T f^n\left(s, \mathbb{E}^{\mathscr{F}_s}[\xi], Z_s^{n,1}\right)ds - \int_t^T Z_s^{n,1}\,dW_s, \qquad t \in [T_{n-1}, T]. \tag{3.14}$$

The generator of BSDE (3.14) is

$$f^{n,1}(s, z) := f^n\left(s, \mathbb{E}^{\mathscr{F}_s}[\xi], z\right),$$

which does not depend on y and is uniformly Lipschitz continuous in z. Moreover, we have

$$\mathbb{E}\left[\int_{T_{n-1}}^T \left|f^{n,1}(s, 0)\right|^{p_0} ds\right] < \infty$$

due to the linear growth property of $f^n(t, \cdot, 0)$. Thus BSDE (3.14) has a unique solution $(Y^{n,1}, Z^{n,1})$ in $\mathcal{S}^{p_0,1}(\mathbb{R}^k)\times\mathcal{H}^{p_0,1}(\mathbb{R}^{k\times d})$ (see [21, Theorem 5.1]).

In a similar way, we show that the following BSDE has a unique solution $(Y^{n,2}, Z^{n,2})$ in $\mathcal{S}^{p_0,2}(\mathbb{R}^k) \times \mathcal{H}^{p_0,2}(\mathbb{R}^{k\times d})$:

$$\begin{aligned} Y_t^{n,2} = Y_{T_{n-1}}^{n,1} &+ \int_t^{T_{n-1}} f^n\left(s, \mathbb{E}^{\mathscr{F}_s}\left[Y_{T_{n-1}}^{n,1}\right], Z_s^{n,2}\right)ds \\ &- \int_t^{T_{n-1}} Z_s^{n,2}\,dW_s, \quad t \in [T_{n-2}, T_{n-1}]. \end{aligned}$$

Inductively in a backward way, it follows that for each $i = 1, \dots, n$, the following BSDE

$$\begin{aligned} Y_t^{n,i} = Y_{T_{n-i+1}}^{n,i-1} &+ \int_t^{T_{n-i+1}} f^n\left(s, \mathbb{E}^{\mathscr{F}_s}\left[Y_{T_{n-i+1}}^{n,i-1}\right], Z_s^{n,i}\right)ds \\ &- \int_t^{T_{n-i+1}} Z_s^{n,i}\,dW_s, \quad t \in [T_{n-i}, T_{n-i+1}], \end{aligned}$$

has a unique solution $(Y^{n,i}, Z^{n,i})$ in $\mathcal{S}^{p_0,i}(\mathbb{R}^k) \times \mathcal{H}^{p_0,i}(\mathbb{R}^{k\times d})$, where $Y_T^{n,0} := \xi$.

We define

$$Y_t^n := \sum_{i=1}^n Y_t^{n,i} \cdot \chi_{[T_{n-i}, T_{n-i+1})}(t) + \xi \cdot \chi_{\{T\}}(t)$$

and

$$Z_t^n := \sum_{i=1}^n Z_t^{n,i} \cdot \chi_{[T_{n-i}, T_{n-i+1})}(t) + Z_T^{n,1} \cdot \chi_{\{T\}}(t).$$

Thus, (Y^n, Z^n) lies in $\mathcal{S}_T^{p_0}(\mathbb{R}^k) \times \mathcal{H}_T^{p_0}(\mathbb{R}^{k\times d})$ and is the unique solution of the following BSDE

$$Y_t^n = \xi + \int_t^T f^n\left(s, \mathbb{E}^{\mathscr{F}_s}\left[Y_{\varphi(n,s)}^n\right], Z_s^n\right)ds - \int_t^T Z_s^n\,dW_s, \quad t \in [0, T]. \tag{3.15}$$

For the approximated solution $\{(Y^n, Z^n)\}_{n\ge1}$ defined by (3.15), we have the following convergence result.

Theorem 3.2. *Let Assumptions* (**Y1**)–(**Y5**) *be satisfied with* $p_0 > 2(2l + 1)$. *Then for any* $p \in [1, \frac{p_0}{2l+1})$, *we have*

$$\left\|Y^n - Y\right\|_{\mathcal{S}_T^p(\mathbb{R}^k)} + \left\|Z^n - Z\right\|_{\mathcal{H}_T^p(\mathbb{R}^{k\times d})} \le \frac{C}{\sqrt{n}},$$

where C is a positive constant independent with n.

Remark 3.1. For the special case that the generator $f := f(t, y)$ is independent with z, we could give an explicit formula for the approximation solution

$$Y_t^n := \mathbb{E}^{\mathscr{F}_t}\left[\xi + \int_t^T f^n\left(s, \mathbb{E}^{\mathscr{F}_s}\left[Y_{\varphi(n,s)}^n\right]\right)ds\right]$$

in an inductively backward way, which has the half-order accuracy $\|Y^n - Y\|_{\mathcal{S}_T^p(\mathbb{R}^k)} \leq \frac{C}{\sqrt{n}}$.

4. Proofs

4.1. Some auxiliary results

The following lemma is an extended dominated convergence theorem (see Yan and Liu [22, page 130]).

Lemma 4.1. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a series of random variables and $\{U_n\}_{n\in\mathbb{N}}$ a series of integrable non-negative random variables. If*

(i) $|X_n| \leq U_n$ and $X_n \longrightarrow X$ a.s.;
(ii) There is an integrable random variable U such that $U_n \longrightarrow U$ a.s., and $\int U_n \longrightarrow \int U$.

Then we have $\int |X_n - X| \longrightarrow 0$. In particular, we have $\int X_n \longrightarrow \int X$.

From the dominated convergence theorem and the BDG inequality, we have the following lemma.

Lemma 4.2. *For $p \geq 2$, if $(Y, Z) \in \mathcal{S}_T^p(\mathbb{R}^k) \times \mathcal{H}_T^p(\mathbb{R}^{k\times d})$, then*

$$\left\{ \int_0^t |Y_s|^{p-2} \langle Y_s, Z_s dW_s \rangle, \quad 0 \leq t \leq T \right\}$$

is a martingale.

The following lemma plays a crucial role in the proof of Theorem 3.1, and can be found in Hu and Tang [7, Lemma A.4].

Lemma 4.3. *For $\widetilde{K} > 0$, there are two positive constants c_1 and c_2 depending only on $\widetilde{K}$ such that for any BMO martingale M, we have*

$$c_1 \|M\|_{BMO(\mathbb{P})} \leq \|\widetilde{M}\|_{BMO(\widetilde{\mathbb{P}})} \leq c_2 \|M\|_{BMO(\mathbb{P})},$$

for any one-dimensional BMO martingale N with $\|N\|_{BMO(\mathbb{P})} \leq \widetilde{K}$, where $\widetilde{M} := M - \langle M, N\rangle$ and $d\widetilde{\mathbb{P}} := \mathcal{E}(N)_0^T \, d\mathbb{P}$.

The following lemma is used to prove Theorem 3.2, and is available in Gyöngy and Krylov [23, Lemma 3.2].

Lemma 4.4. *Let $T \in (0,\infty)$, $f := \{f_t\}_{0\leq t\leq T}$ and $g := \{g_t\}_{0\leq t\leq T}$ be non-negative continuous $\mathscr{F}$-adapted processes such that, for any constant $c > 0$,*

$$\mathbb{E}\left[f_\tau \chi_{g_0\leq c}\right] \leq \mathbb{E}\left[g_\tau \chi_{g_0 \leq c}\right],$$

for any stopping time $\tau \leq T$. Then, for any stopping time $\sigma \leq T$ and $\gamma \in (0, 1)$, we have

$$\mathbb{E}\left[\sup_{0\leq t\leq\sigma} f_t^\gamma\right] \leq \frac{2-\gamma}{1-\gamma}\mathbb{E}\left[\sup_{0\leq t\leq\sigma} g_t^\gamma\right].$$

4.2. The proof of Theorem 3.1 .

The proof consists of the following four steps.

Step 1: We give a prior estimation for $\sup_{n\in\mathbb{N}}\left\{\|Y^n\|_{\mathcal{S}_T^\infty(\mathbb{R})}\right\}$.

Fix $n \in \mathbb{N}$. According to (3.6), for any $i = 1, 2, \ldots, n$ we have

$$\begin{aligned}
\|Y^{n,i}\|_{\mathcal{S}^{\infty,i}(\mathbb{R})} &\leq \left(\alpha + \frac{\gamma}{2}\right)\frac{T}{n} + \left(1 + \frac{\beta T}{n}\right)\|Y^{n,i-1}\|_{\mathcal{S}^{\infty,i-1}(\mathbb{R})} \\
&\leq \left(\alpha + \frac{\gamma}{2}\right)\frac{T}{n} + \left(1 + \frac{\beta T}{n}\right)\left[\left(\alpha + \frac{\gamma}{2}\right)\frac{T}{n} + \left(1+\frac{\beta T}{n}\right)\|Y^{n,i-2}\|_{\mathcal{S}^{\infty,i-2}(\mathbb{R})}\right] \\
&= \left(\alpha + \frac{\gamma}{2}\right)\frac{T}{n}\left[1 + \left(1+\frac{\beta T}{n}\right)\right] + \left(1+\frac{\beta T}{n}\right)^2 \|Y^{n,i-2}\|_{\mathcal{S}^{\infty,i-2}(\mathbb{R})} \\
&\leq \cdots \\
&\leq \left(\alpha + \frac{\gamma}{2}\right)\frac{T}{n}\left[1 + \left(1+\frac{\beta T}{n}\right) + \cdots + \left(1+\frac{\beta T}{n}\right)^{i-1}\right] \\
&\quad + \left(1+\frac{\beta T}{n}\right)^i \|\xi\|_{\mathcal{L}^\infty(\mathbb{R})}.
\end{aligned}$$

If $\beta = 0$, it is obvious that

$$\|Y^{n,i}\|_{\mathcal{S}^{\infty,i}(\mathbb{R})} \leq \left(\alpha + \frac{\gamma}{2}\right)T + \|\xi\|_{\mathcal{L}^\infty(\mathbb{R})} := M_0.$$

If $\beta \neq 0$, we have

$$\begin{aligned}
\|Y^{n,i}\|_{\mathcal{S}^{\infty,i}(\mathbb{R})} &\leq \left(\alpha + \frac{\gamma}{2}\right)\frac{T}{n}\left[1 + \cdots + \left(1+\frac{\beta T}{n}\right)^{n-1}\right] \\
&\quad + \left(1+\frac{\beta T}{n}\right)^n \|\xi\|_{\mathcal{L}^\infty(\mathbb{R})} \\
&\leq \frac{1}{\beta}\left(\alpha + \frac{\gamma}{2}\right)\left[\left(1+\frac{\beta T}{n}\right)^n - 1\right] + e^{\beta T}\|\xi\|_{\mathcal{L}^\infty(\mathbb{R})} \\
&\leq \frac{1}{\beta}\left(\alpha + \frac{\gamma}{2}\right)\left(e^{\beta T} - 1\right) + e^{\beta T}\|\xi\|_{\mathcal{L}^\infty(\mathbb{R})} := \overline{M}.
\end{aligned}$$

Set $M := \max\{M_0, \overline{M}\}$. Then we obtain that

$$\sup_{n\in\mathbb{N}}\left\{\|Y^n\|_{\mathcal{S}_T^\infty(\mathbb{R})}\right\} \leq M.$$

Step 2: We prove that $\{Z^n \circ W\}_{n\geq 1} \subseteq BMO$ are uniformly bounded, which leads to the fact that the local error tends to *zero* uniformly in $[0, T] \times \Omega$ when n goes to ∞. To do this, we define

$$\phi(y) := \gamma^{-2}\left[\exp(\gamma|y|) - \gamma|y| - 1\right], \qquad y \in \mathbb{R}.$$

One can verify that

$$\phi'(y) = \gamma^{-1}\left[\exp(\gamma|y|) - 1\right]\operatorname{sgn}(y), \quad \phi''(y) = \exp(\gamma|y|),$$
$$\phi''(y) - \gamma|\phi'(y)| = 1.$$

For any bounded stopping time τ, using Itô's formula to compute $\phi(Y_\tau^n)$, we have

$$\begin{aligned}
&\phi(Y_\tau^n) + \frac{1}{2}\mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^T \phi''(Y_s^n)|Z_s^n|^2\, ds\right] \\
&= \mathbb{E}^{\mathscr{F}_\tau}\left[\phi(\xi)\right] + \mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^T \phi'(Y_s^n) f(s, \mathbb{E}^{\mathscr{F}_s}\left[Y^n_{\varphi(n,s)}\right], Z_s^n)\, ds\right] \\
&\leq \phi(\|\xi\|_{\mathcal{L}_T^\infty(\mathbb{R})}) + \mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^T |\phi'(Y_s^n)|\left(\alpha + \frac{\gamma}{2} + \beta M + \frac{\gamma}{2}|Z_s^n|^2\right) ds\right].
\end{aligned}$$

It holds that

$$\begin{aligned}
\phi(Y_\tau^n) + \frac{1}{2}\mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^T |Z_s^n|^2 ds\right] &\leq \phi(\|\xi\|_{\mathcal{L}_T^\infty(\mathbb{R})}) \\
&\quad + \mathbb{E}^{\mathscr{F}_\tau}\left[\int_\tau^T |\phi'(Y_s^n)|\left(\alpha + \frac{\gamma}{2} + \beta M\right) ds\right] \\
&\leq \phi(M) + |\phi'(M)|\left(\alpha + \frac{\gamma}{2} + \beta M\right)T := \frac{N^2}{2} < \infty.
\end{aligned}$$

Thus, we have

$$\sup_{n\in\mathbb{N}} \|Z^n \circ W\|_{BMO} \leq N.$$

According to (3.7), we have

$$Y_t^n = \xi + \int_t^T f\left(s, \mathbb{E}^{\mathscr{F}_s}\left[Y^n_{\varphi(n,s)}\right], Z_s^n\right) ds - \int_t^T Z_s^n\, dW_s, \quad t \in [0, T]. \tag{4.1}$$

Define the local error

$$p_t^n := \mathbb{E}^{\mathscr{F}_t}\left[Y^n_{\varphi(n,t)}\right] - Y_t^n, \quad t \in [0, T].$$

Then BSDE (4.1) reads

$$\begin{cases} dY_t^n &= -f\left(t, Y_t^n + p_t^n, Z_t^n\right) dt + Z_t^n \, dW_t, \quad t \in [0,T); \\ Y_T^n &= \xi. \end{cases} \tag{4.2}$$

Since $Z^n \circ W$ is a BMO martingale, we have

$$p_t^n = -\mathbb{E}^{\mathscr{F}_t}\left[\int_t^{\varphi(n,t)} f\left(s, \mathbb{E}^{\mathscr{F}_s}\left[Y^n_{\varphi(n,s)}\right], Z_s^n\right) ds\right], \quad t \in [0,T]. \tag{4.3}$$

It yields that

$$\begin{aligned} |p_t^n| &\le \mathbb{E}^{\mathscr{F}_t}\left[\int_t^{\varphi(n,t)} \left|f\left(s, \mathbb{E}^{\mathscr{F}_s}\left[Y^n_{\varphi(n,s)}\right], Z_s^n\right)\right| ds\right] \\ &\le (\alpha+\beta M)\frac{T}{n} + \frac{\gamma}{2}\mathbb{E}^{\mathscr{F}_t}\left[\int_t^{\varphi(n,t)} \left|Z_s^n\right|^{2-\varepsilon} ds\right] \\ &\le (\alpha+\beta M)\frac{T}{n} + \frac{\gamma}{2}\mathbb{E}^{\mathscr{F}_t}\left[\left(\int_t^{\varphi(n,t)} \left|Z_s^n\right|^2 ds\right)^{1-\frac{\varepsilon}{2}} (\varphi(n,t)-t)^{\frac{\varepsilon}{2}}\right] \\ &\le (\alpha+\beta M)\frac{T}{n} + \frac{\gamma}{2}(N^2+1)\left(\frac{T}{n}\right)^{\frac{\varepsilon}{2}} := \delta_n. \end{aligned} \tag{4.4}$$

Note that $\{\delta_n\}_{n\in\mathbb{N}}$ is a sequence of deterministic non-negative numbers with $\delta_n \le Cn^{-\frac{\varepsilon}{2}}$.

Step 3: We prove that $\{(Y^n, Z^n \circ W)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}_T^\infty(\mathbb{R}) \times BMO$. Define $\psi(t) := e^{-2K(T-t)}$. For each $n, m \in \mathbb{N}$, using Itô's formula, we have

$$\begin{aligned} &\psi_\tau |Y_\tau^n - Y_\tau^m|^2 + \int_\tau^T \psi_s |Z_s^n - Z_s^m|^2 \, ds \\ &+2\int_\tau^T \psi_s (Y_s^n - Y_s^m)(Z_s^n - Z_s^m)\, dW_s \\ =\ & 2\int_\tau^T \psi_s (Y_s^n - Y_s^m)\left[f(s, Y_s^n + p_s^n, Z_s^n) - f(s, Y_s^m + p_s^m, Z_s^m)\right] ds \\ &-2K\int_\tau^T \psi_s |Y_s^n - Y_s^m|^2 ds \\ =\ & 2\int_\tau^T \psi_s (Y_s^n - Y_s^m)\left[f(s, Y_s^n + p_s^n, Z_s^n) - f(s, Y_s^n + p_s^n, Z_s^m)\right] ds \\ &-2K\int_\tau^T \psi_s |Y_s^n - Y_s^m|^2 ds \\ &+2\int_\tau^T \psi_s \left[(Y_s^n + p_s^n) - (Y_s^m + p_s^m)\right]\left[f(s, Y_s^n + p_s^n, Z_s^m)\right. \\ &\left. -f(s, Y_s^m + p_s^m, Z_s^m)\right] ds \\ &-2\int_\tau^T \psi_s (p_s^n - p_s^m)\left[f(s, Y_s^n + p_s^n, Z_s^m) - f(s, Y_s^m + p_s^m, Z_s^m)\right] ds, \end{aligned}$$

where τ is an arbitrary bounded stopping time. Furthermore, we have

$$\begin{aligned} &\psi_\tau |Y_\tau^n - Y_\tau^m|^2 + \int_\tau^T \psi_s |Z_s^n - Z_s^m|^2 \, ds \\ &+2\int_\tau^T \psi_s (Y_s^n - Y_s^m)(Z_s^n - Z_s^m)\, dW_s \\ \le\ & K\int_\tau^T \psi_s |(Y_s^n + p_s^n) - (Y_s^m + p_s^m)|^2 ds - 2K\int_\tau^T \psi_s |Y_s^n - Y_s^m|^2 ds \\ &+4\int_\tau^T \psi_s |p_s^n - p_s^m| \left(\alpha + \beta M + \frac{\gamma}{2} + \frac{\gamma}{2}|Z_s^m|^2\right) ds \\ &+2\int_\tau^T \psi_s (Y_s^n - Y_s^m)\left[f(s, Y_s^n + p_s^n, Z_s^n) - f(s, Y_s^n + p_s^n, Z_s^m)\right] ds \\ \le\ & 2K\int_\tau^T \psi_s |p_s^n - p_s^m|^2 ds + 4\int_\tau^T \psi_s (|p_s^n| + |p_s^m|) \\ &\times \left(\alpha + \beta M + \frac{\gamma}{2} + \frac{\gamma}{2}|Z_s^m|^2\right) ds \\ &+2\int_\tau^T \psi_s (Y_s^n - Y_s^m)\left[f(s, Y_s^n + p_s^n, Z_s^n) - f(s, Y_s^n + p_s^n, Z_s^m)\right] ds \\ \le\ & 4KT(\delta_n^2 + \delta_m^2) + 4T\left(\alpha + \beta M + \frac{\gamma}{2}\right)(\delta_n + \delta_m) + 2\gamma(\delta_n + \delta_m) \\ &\times \int_\tau^T |Z_s^m|^2 ds \\ &+2\int_\tau^T \psi_s (Y_s^n - Y_s^m)\left[f(s, Y_s^n + p_s^n, Z_s^n) - f(s, Y_s^n + p_s^n, Z_s^m)\right] ds. \end{aligned}$$

For any fixed $n, m \in \mathbb{N}$, we define $L_s^{n,m} = \left((L_s^{n,m})_1, \ldots, (L_s^{n,m})_i, \ldots, (L_s^{n,m})_d\right)^{\mathrm{T}}$ as follows:

$$(L_s^{n,m})_i := \begin{cases} \dfrac{f(s, Y_s^n + p_s^n, \bar{Z}_i^{n,m}(s)) - f(s, Y_s^n + p_s^n, \bar{Z}_{i+1}^{n,m}(s))}{(Z_s^n - Z_s^m)_i} & \text{if } (Z_s^n - Z_s^m)_i \neq 0, \\ 0 & \text{if } (Z_s^n - Z_s^m)_i = 0, \end{cases}$$

where $\bar{Z}_i^{n,m}(s) := \left((Z_s^m)_1, \ldots, (Z_s^m)_{i-1}, (Z_s^n)_i, (Z_s^n)_{i+1}, \ldots, (Z_s^n)_d\right)^{\mathrm{T}}$. Then we have

$$f(s, Y_s^n + p_s^n, Z_s^n) - f(s, Y_s^n + p_s^n, Z_s^m) = \langle Z_s^n - Z_s^m, L_s^{n,m}\rangle.$$

By virtue of (3.2), we get

$$|L_s^{n,m}| \le C(1 + \rho(M) + |Z_s^n| + |Z_s^m|).$$

Thus $L^{n,m} \circ W \in BMO$. Moreover, there is a uniform bound for $\{L^{n,m} \circ W\}_{n,m\ge 1}$ in BMO,

$$\sup_{n,m\in\mathbb{N}} \|L^{n,m} \circ W\|_{BMO} \le \widetilde{K},$$

where $\widetilde{K} := CT\left[1 + \rho(M)\right] + 2CN$. Thus

$$\left\{W_t^{n,m} := W_t - \int_0^t L_s^{n,m}\, ds, 0 \le t \le T\right\}$$

is a standard Brownian Motion with respect to the equivalent probability measure $\mathbb{P}^{n,m}$ defined by

$$d\mathbb{P}^{n,m} := \mathcal{E}(L^{n,m} \circ W)_0^T \, d\mathbb{P}.$$

It follows that

$$\begin{aligned} &\psi_0 |Y_\tau^n - Y_\tau^m|^2 + \psi_0 \int_\tau^T |Z_s^n - Z_s^m|^2 \, ds \\ &+2\int_\tau^T \psi_s (Y_s^n - Y_s^m)(Z_s^n - Z_s^m)\, dW_s^{n,m} \\ \le\ & 4KT(\delta_n^2 + \delta_m^2) + 4T(\alpha + \beta M + \frac{\gamma}{2})(\delta_n + \delta_m) \\ &+2\gamma(\delta_n + \delta_m)\int_\tau^T |Z_s^m|^2 \, ds. \end{aligned}$$

Denote by $\mathbb{E}_{n,m}^{\mathscr{F}_\tau}[\ \cdot\ ]$ the expectation operator with respect to the probability measure $\mathbb{P}^{n,m}$, conditioned on $\mathscr{F}_\tau$. Thus

$$\begin{aligned} &\psi_0 |Y_\tau^n - Y_\tau^m|^2 + \psi_0 \mathbb{E}_{n,m}^{\mathscr{F}_\tau}\left[\int_\tau^T |Z_s^n - Z_s^m|^2 \, ds\right] \\ \le\ & 4KT(\delta_n^2 + \delta_m^2) + 4T(\alpha + \beta M + \frac{\gamma}{2})(\delta_n + \delta_m) \\ &+2\gamma(\delta_n + \delta_m)\mathbb{E}_{n,m}^{\mathscr{F}_\tau}\left[\int_\tau^T |Z_s^m|^2 \, ds\right]. \end{aligned}$$

This implies that

$$\begin{aligned}&\left\|Y^n-Y^m\right\|^2_{\mathcal{S}^\infty_T(\mathbb{R})}+\left\|Z^n\circ W^{n,m}-Z^m\circ W^{n,m}\right\|^2_{BMO(\mathbb{P}^{n,m})}\\ \le&4KTe^{2KT}\left(\delta_n^2+\delta_m^2\right)+2e^{2KT}\Big[2T\left(\alpha+\beta M+\frac{\gamma}{2}\right)\\ &+\gamma\|Z^m\circ W^{n,m}\|^2_{BMO(\mathbb{P}^{n,m})}\Big](\delta_n+\delta_m).\end{aligned}$$

By Lemma 4.3, there are two constants $c_1>0$ and $c_2>0$ depending only on $\widetilde{K}$ such that

$$\begin{aligned}&\|Y^n-Y^m\|^2_{\mathcal{S}^\infty_T(\mathbb{R})}+c_1^2\|Z^n\circ W-Z^m\circ W\|^2_{BMO(\mathbb{P})}\\ \le&\ 4KTe^{2KT}(\delta_n^2+\delta_m^2)+2e^{2KT}\Big[2T\left(\alpha+\beta M+\frac{\gamma}{2}\right)\\ &+\gamma c_2^2\|Z^m\circ W\|^2_{BMO(\mathbb{P}^{n,m})}\Big](\delta_n+\delta_m)\\ \le&\ 4KTe^{2KT}(\delta_n^2+\delta_m^2)+2e^{2KT}\Big[2T\left(\alpha+\beta M+\frac{\gamma}{2}\right)\\ &+\gamma c_2^2N^2\Big](\delta_n+\delta_m).\end{aligned}\tag{4.5}$$

Thus $(Y^n,Z^n\circ W)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}^\infty_T(\mathbb{R})\times BMO$. There exists $(Y,Z\circ W)\in\mathcal{S}^\infty_T(\mathbb{R})\times BMO$ such that

$$\|Y^n-Y\|_{\mathcal{S}^\infty_T(\mathbb{R})}+\|Z^n\circ W-Z\circ W\|_{BMO}\le\delta_n^{\frac12}\le\frac{C}{n^{\frac{\varepsilon}{4}}}.$$

Step 4: Finally, we take limits of Eq. (4.2). For any $t\in[0,T]$, we have

$$\begin{aligned}&Y_t-\left(\xi+\int_t^T f(s,Y_s,Z_s)\,ds-\int_t^T Z_s\,dW_s\right)\\ =&\ (Y_t-Y_t^n)+\int_t^T\left[f(s,Y_s^n+p_s^n,Z_s^n)-f(s,Y_s,Z_s)\right]ds\\ &+\int_t^T(Z_s-Z_s^n)\,dW_s,\end{aligned}$$

which leads to that

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T}\left|Y_t-\left(\xi+\int_t^T f(s,Y_s,Z_s)\,ds-\int_t^T Z_s\,dW_s\right)\right|\right]\\ \le&\ \|Y^n-Y\|_{\mathcal{S}^\infty_T(\mathbb{R})}+\mathbb{E}\left[\int_0^T\left|f(s,Y_s^n+p_s^n,Z_s^n)-f(s,Y_s,Z_s)\right|ds\right]\\ &+C\|Z^n-Z\|_{\mathcal{H}^1_T(\mathbb{R}^d)}.\end{aligned}\tag{4.6}$$

It is easy to see that the first term and the third term in the right-hand side of (4.6) tend to *zero* when n goes to ∞ because of the conclusions in step 3. For the third term, we can extract subsequences of $(Y^n)_{n\in\mathbb{N}}$, $(p^n)_{n\in\mathbb{N}}$ and $(Z^n)_{n\in\mathbb{N}}$ (for the sake of simplicity of notations, we still denote these subsequences by $(Y^n)_{n\in\mathbb{N}}$, $(p^n)_{n\in\mathbb{N}}$ and $(Z^n)_{n\in\mathbb{N}}$ respectively) such that

$$\lim_{n\to\infty}Y_t^n=Y_t,\quad\lim_{n\to\infty}Z_t^n=Z_t,\quad\lim_{n\to\infty}p_t^n=0,\qquad a.s.\quad a.e.$$

We define

$$F_s^n:=|f(s,Y_s^n+p_s^n,Z_s^n)-f(s,Y_s,Z_s)|.$$

It turns out that $\lim_{n\to\infty}F_s^n=0$, a.s., a.e. since f is continuous in the last two arguments (y,z). Note that

$$\begin{aligned}|F_s^n|\le&\ |f(s,Y_s^n+p_s^n,Z_s^n)|+|f(s,Y_s,Z_s)|\\ \le&\ \alpha+\frac{\gamma}{2}+\beta M+\frac{\gamma}{2}|Z_s^n|^2+\alpha+\frac{\gamma}{2}\\ &+\beta\|Y\|_{\mathcal{S}^\infty_T(\mathbb{R})}+\frac{\gamma}{2}|Z_s|^2\le G_s^n,\end{aligned}$$

with

$$G_s^n:=2\alpha+\gamma+\beta(M+\|Y\|_{\mathcal{S}^\infty_T(\mathbb{R})})+\frac{\gamma}{2}(|Z_s|^2+|Z_s^n|^2).$$

Define

$$G_s:=2\alpha+\gamma+\beta(M+\|Y\|_{\mathcal{S}^\infty_T(\mathbb{R})})+\gamma|Z_s|^2.$$

We have

(i) $|F_s^n|\le G_s^n$, a.s., a.e.;
(ii) For any $n\in\mathbb{N}$, G^n is integrable with respect to $d\mathbb{P}\otimes dt$; $\lim_{n\to\infty}G_s^n=G_s$, a.s., a.e.;
(iii) $\lim_{n\to\infty}\mathbb{E}\left[\int_0^T G_s^n\,ds\right]=\mathbb{E}\left[\int_0^T G_s\,ds\right]$.

According to Lemma 4.1, we know that

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}\left[\int_0^T|f(s,Y_s^n+p_s^n,Z_s^n)-f(s,Y_s,Z_s)|\,ds\right]\\ &=\mathbb{E}\left[\int_0^T\lim_{n\to\infty}F_s^n\,ds\right]=0,\end{aligned}$$

which implies that

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|Y_t-\left(\xi+\int_t^T f(s,Y_s,Z_s)\,ds-\int_t^T Z_s\,dW_s\right)\right|\right]=0.$$

Therefore, we have

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s)\,ds-\int_t^T Z_s\,dW_s.$$

The existence of solution is proved.

The uniqueness of solution is an immediate consequence of (4.5). In fact, it is sufficient to let $(Y^n,Z^n\circ W)$ be the solution $(Y,Z\circ W)$ if n is even and be another solution $(Y',Z'\circ W)$ if n is odd with $p^n\equiv 0$. The proof is complete. □

Remark 4.1. According to (4.4), we see that the sub-quadratic assumption $\varepsilon>0$ is essential to prove that the local error p^n tends to *zero* uniformly in $[0,T]\times\Omega$ when n goes to ∞.

4.3. The proof of Theorem 3.2

The proof consists of the following three steps.

Step 1: We give an a priori estimate for $\sup_{n\in\mathbb{N}}\left\{\sup_{0\le t\le T}\mathbb{E}\left[|Y_t^n|^{p_0}\right]\right\}$.

Fix $n\in\mathbb{N}$. Due to $p_0\ge 2(2l+1)\ge 2$, we can use Itô's formula to get that

$$\begin{aligned}&|Y_t^n|^{p_0}+\frac{p_0(p_0-1)}{2}\int_t^T|Y_s^n|^{p_0-2}|Z_s^n|^2\,ds\\ =&\ |\xi|^{p_0}+p_0\int_t^T|Y_s^n|^{p_0-2}\left\langle Y_s^n,f^n(s,\mathbb{E}^{\mathscr{F}_s}\left[Y^n_{\phi(n,s)}\right],Z_s^n)\right\rangle ds\\ &-p_0\int_t^T|Y_s^n|^{p_0-2}\left\langle Y_s^n,Z_s^n dW_s\right\rangle.\end{aligned}$$

Since $(Y^n,Z^n)\in\mathcal{S}^{p_0}_T(\mathbb{R}^k)\times\mathcal{H}^{p_0}_T(\mathbb{R}^{k\times d})$, using Lemma 4.2, we know that the process

$$\left\{\int_0^t|Y_s^n|^{p_0-2}\left\langle Y_s^n,Z_s^n dW_s\right\rangle,\quad 0\le t\le T\right\}$$

is a martingale. Thus, we have

$$\begin{aligned}&|Y_t^n|^{p_0}+\frac{p_0(p_0-1)}{2}\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T|Y_s^n|^{p_0-2}|Z_s^n|^2\,ds\right]\\ =&\ \mathbb{E}^{\mathscr{F}_t}\left[|\xi|^{p_0}\right]+p_0\mathbb{E}^{\mathscr{F}_t}\Big[\int_t^T|Y_s^n|^{p_0-2}\langle Y_s^n,f^n(s,\mathbb{E}^{\mathscr{F}_s}\left[Y^n_{\varphi(n,s)}\right],Z_s^n)\\ &-f^n(s,\mathbb{E}^{\mathscr{F}_s}\left[Y^n_{\varphi(n,s)}\right],0)\rangle\,ds\Big]\\ &+p_0\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T|Y_s^n|^{p_0-2}\langle\mathbb{E}^{\mathscr{F}_s}\left[Y^n_{\varphi(n,s)}\right],f^n(s,\mathbb{E}^{\mathscr{F}_s}\left[Y^n_{\varphi(n,s)}\right],0)\rangle\,ds\right]\end{aligned}$$

$$
\begin{aligned}
&+p_0\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T |Y_s^n|^{p_0-2}\langle \mathbb{E}^{\mathscr{F}_s}\left[Y_s^n - Y_{\varphi(n,s)}^n\right], f^n(s,\mathbb{E}^{\mathscr{F}_s}\left[Y_{\varphi(n,s)}^n\right],0)\rangle\, ds\right]\\
&\le \mathbb{E}^{\mathscr{F}_t}\left[|\xi|^{p_0}\right] + p_0\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T |Y_s^n|^{p_0-2}L|Y_s^n||Z_s^n|ds\right]\\
&+C\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T |Y_s^n|^{p_0-2}\left(1+\left|\mathbb{E}^{\mathscr{F}_s}\left[Y_{\varphi(n,s)}^n\right]\right|^2\right) ds\right]\\
&+p_0\mathbb{E}^{\mathscr{F}_t}\Bigg[\int_t^T |Y_s^n|^{p_0-2}\mathbb{E}^{\mathscr{F}_s}\Bigg[\int_s^{\varphi(n,s)} |f^n(r,\mathbb{E}^{\mathscr{F}_r}\left[Y_{\varphi(n,r)}^n\right],Z_r^n)|dr\Bigg]\\
&\times \left|f^n(s,\mathbb{E}^{\mathscr{F}_s}\left[Y_{\varphi(n,s)}^n\right],0)\right| ds\Bigg].
\end{aligned}
$$

By Young's inequality, we have

$$
\begin{aligned}
&|Y_t^n|^{p_0} + \frac{p_0(p_0-1)}{2}\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T |Y_s^n|^{p_0-2}|Z_s^n|^2\, ds\right]\\
&\le \mathbb{E}^{\mathscr{F}_t}\left[|\xi|^{p_0}\right] + p_0\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T |Y_s^n|^{p_0-2}\left(\frac{L^2}{p_0-1}|Y_s^n|^2 + \frac{p_0-1}{4}|Z_s^n|^2\right) ds\right]\\
&+C\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T (1+|Y_s^n|^{p_0}+\mathbb{E}^{\mathscr{F}_s}\left[|Y_{\varphi(n,s)}^n|^{p_0}\right]) ds\right]\\
&+C\mathbb{E}^{\mathscr{F}_t}\Bigg[\int_t^T |Y_s^n|^{p_0-2}\int_s^{\varphi(n,s)}\Bigg\{n^{2\alpha}\left(1+\left|\mathbb{E}^{\mathscr{F}_s}[Y_{\varphi(n,s)}^n]\right|^2\right)\\
&+\frac{p_0(p_0-1)}{8CT}\mathbb{E}^{\mathscr{F}_s}\left[|Z_r^n|^2\right]\Bigg\}\, drds\Bigg]\\
&\le \mathbb{E}^{\mathscr{F}_t}\left[|\xi|^{p_0}\right] + \frac{p_0(p_0-1)}{4}\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T |Y_s^n|^{p_0-2}|Z_s^n|^2\, ds\right]\\
&+\frac{p_0(p_0-1)}{8T}\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T |Y_s^n|^{p_0-2}\mathbb{E}^{\mathscr{F}_s}\left[\int_s^{\varphi(n,s)} \left|Z_r^n\right|^2 dr\right] ds\right]\\
&+C\left(1+n^{2\alpha-1}\right)\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T (1+|Y_s^n|^{p_0}+\mathbb{E}^{\mathscr{F}_s}\left[|Y_{\varphi(n,s)}^n|^{p_0}\right]) ds\right], \qquad (4.7)
\end{aligned}
$$

for all $t\in[0,T]$.

We set

$$2\alpha - 1 \le 0. \qquad (4.8)$$

Temporarily, we let $p_0 = 2$. It follows

$$
\begin{aligned}
\frac{1}{2}\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T |Z_s^n|^2\, ds\right] &\le \mathbb{E}^{\mathscr{F}_t}\left[|\xi|^2\right] + \frac{1}{4T}\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T\int_s^{\varphi(n,s)} \left|Z_r^n\right|^2 drds\right]\\
&+C\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T \left(1+|Y_s^n|^2+|Y_{\varphi(n,s)}^n|^2\right) ds\right].
\end{aligned}
$$

Integrating by part, we have

$$
\begin{aligned}
\int_t^T\int_s^{\varphi(n,s)} \left|Z_r^n\right|^2 drds &\le \int_t^T\int_s^T \left|Z_r^n\right|^2 drds = \int_t^T (s-t)\left|Z_s^n\right|^2 ds\\
&\le T\int_t^T \left|Z_s^n\right|^2 ds.
\end{aligned}
$$

It yields that

$$
\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T |Z_s^n|^2 ds\right] \le C\mathbb{E}^{\mathscr{F}_t}\left[|\xi|^2\right] + C\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T \left(1+|Y_s^n|^2+|Y_{\varphi(n,s)}^n|^2\right) ds\right]. \qquad (4.9)
$$

Now we let $p_0 \ge 2(2l+1)$. According to (4.7), we have

$$
\begin{aligned}
\mathbb{E}\left[|Y_t^n|^{p_0}\right] &\le C + C\int_t^T \left(\mathbb{E}\left[|Y_s^n|^{p_0}\right] + \mathbb{E}\left[|Y_{\varphi(n,s)}^n|^{p_0}\right]\right) ds\\
&\le C + C\int_t^T \sup_{s\le r\le T}\mathbb{E}\left[|Y_r^n|^{p_0}\right] ds.
\end{aligned}
$$

According to the backward Gronwall's inequality, there is a constant $M_Y > 0$ such that

$$\sup_{n\in\mathbb{N}}\sup_{0\le t\le T}\mathbb{E}\left[|Y_t^n|^{p_0}\right] \le M_Y.$$

Step 2: We prove that $\{Y^n\}_{n\ge1}$ converges to Y in $\mathcal{S}_T^p(\mathbb{R}^k)$.

Set $p_1 := \frac{p_0}{2l+1}$ and note that $p_1 > 2$. By Itô's formula, we obtain

$$
\begin{aligned}
&-d|Y_t^n - Y_t|^{p_1} + \frac{p_1(p_1-1)}{2}|Y_t^n - Y_t|^{p_1-2}|Z_t^n - Z_t|^2\, dt\\
&+p_1|Y_t^n - Y_t|^{p_1-2}\left\langle Y_t^n - Y_t, (Z_t^n - Z_t)dW_t\right\rangle\\
&= p_1|Y_t^n - Y_t|^{p_1-2}\left\langle Y_t^n - Y_t, f^n(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t^n) - f(t,Y_t,Z_t)\right\rangle dt\\
&= p_1|Y_t^n - Y_t|^{p_1-2}\left\langle Y_t^n - Y_t, f^n(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t^n) - f^n(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t)\right\rangle dt\\
&+p_1|Y_t^n - Y_t|^{p_1-2}\left\langle Y_t^n - Y_t, f^n(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t) - f(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t)\right\rangle dt\\
&+p_1|Y_t^n - Y_t|^{p_1-2}\left\langle Y_t^n - Y_t, f(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t) - f(t,Y_t^n,Z_t)\right\rangle dt\\
&+p_1|Y_t^n - Y_t|^{p_1-2}\left\langle Y_t^n - Y_t, f(t,Y_t^n,Z_t) - f(t,Y_t,Z_t)\right\rangle dt.
\end{aligned}
$$

According to the Lipschitz condition and monotonicity condition, we have

$$
\begin{aligned}
&-d|Y_t^n - Y_t|^{p_1} + \frac{p_1(p_1-1)}{2}|Y_t^n - Y_t|^{p_1-2}|Z_t^n - Z_t|^2\, dt\\
&+p_1|Y_t^n - Y_t|^{p_1-2}\left\langle Y_t^n - Y_t, (Z_t^n - Z_t)dW_t\right\rangle\\
&\le p_1|Y_t^n - Y_t|^{p_1-2}\left(\frac{p_1-1}{4}|Z_t^n - Z_t|^2 + \frac{L^2}{p_1-1}|Y_t^n - Y_t|^2\right) dt\\
&+p_1|Y_t^n - Y_t|^{p_1-1}\left|f^n(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t) - f(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t)\right| dt\\
&+p_1|Y_t^n - Y_t|^{p_1-1}\left|f(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t) - f(t,Y_t^n,Z_t)\right| dt\\
&+\frac{p_1K}{2}|Y_t^n - Y_t|^{p_1}dt.
\end{aligned}
$$

It holds that

$$
\begin{aligned}
-d|Y_t^n - Y_t|^{p_1} \le &-\frac{p_1(p_1-1)}{4}|Y_t^n - Y_t|^{p_1-2}|Z_t^n - Z_t|^2\, dt\\
&-p_1|Y_t^n - Y_t|^{p_1-2}\left\langle Y_t^n - Y_t, (Z_t^n - Z_t)dW_t\right\rangle\\
&+C|Y_t^n - Y_t|^{p_1}dt\\
&+C|f^n(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t) - f(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t)|^{p_1}dt\\
&+C|f(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t) - f(t,Y_t^n,Z_t)|^{p_1}\, dt. \qquad (4.10)
\end{aligned}
$$

Define

$$\phi(t) := \exp\left[C(t-T)\right], \quad t\in[0,T],$$

where $C > 0$ is the constant in (4.10). Note that $e^{-CT} \le \phi(t) \le 1$. We get

$$
\begin{aligned}
-d(\phi(t)|Y_t^n - Y_t|^{p_1}) &= -C\phi(t)|Y_t^n - Y_t|^{p_1}\, dt - \phi(t)\, d|Y_t^n - Y_t|^{p_1}\\
&\le -\frac{e^{-CT}p_1(p_1-1)}{4}|Y_t^n - Y_t|^{p_1-2}|Z_t^n - Z_t|^2\, dt\\
&\quad -p_1\phi(t)|Y_t^n - Y_t|^{p_1-2}\left\langle Y_t^n - Y_t, (Z_t^n - Z_t)\, dW_t\right\rangle\\
&\quad +C\left|f^n(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t) - f(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t)\right|^{p_1} dt\\
&\quad +C\left|f(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t) - f(t,Y_t^n,Z_t)\right|^{p_1} dt. \qquad (4.11)
\end{aligned}
$$

Since $(Y^n - Y, Z^n - Z) \in \mathcal{S}_T^{p_0}(\mathbb{R}^k)\times\mathcal{H}_T^{p_0}(\mathbb{R}^{k\times d})$, Lemma 4.2 tells us that

$$\left\{\int_0^t \phi(s)|Y_s^n - Y_s|^{p_1-2}\left\langle Y_s^n - Y_s, (Z_s^n - Z_s)dW_s\right\rangle, \quad 0\le t\le T\right\}$$

is a martingale. Then for any stopping time $\tau \le T$, we have

$$\mathbb{E}\left[\phi(\tau)|Y_\tau^n - Y_\tau|^{p_1}\right] \le C\left[M_1(n,p_1) + M_2(n,p_1)\right],$$

where

$$
\begin{aligned}
&M_1(n,p_1) := \mathbb{E}\left[\int_0^T \left|f^n(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t) - f(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],Z_t)\right|^{p_1} dt\right]\\
&= \mathbb{E}\left[\int_0^T \frac{n^{-\alpha p_1}\left|\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right]\right|^{lp_1}}{\left(1+n^{-\alpha}\left|\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right]\right|^l\right)^{p_1}}\left|f(t,\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right],0)\right|^{p_1} dt\right]\\
&\le Cn^{-\alpha p_1}\mathbb{E}\left[\int_0^T \left|\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right]\right|^{lp_1}\left(1+\left|\mathbb{E}^{\mathscr{F}_t}\left[Y_{\varphi(n,t)}^n\right]\right|^{l+1}\right)^{p_1} dt\right]\\
&\le Cn^{-\alpha p_1}\int_0^T \left(1+\mathbb{E}\left[|Y_{\varphi(n,t)}^n|^{p_1(2l+1)}\right]\right) dt \le Cn^{-\alpha p_1},
\end{aligned}
$$

and

$$
\begin{aligned}
&M_2(n,p_1) := \mathbb{E}\left[\int_0^T \left|f(t,\mathbb{E}^{\mathscr{F}_t}\left[Y^n_{\varphi(n,t)}\right],Z_t) - f(t,Y^n_t,Z_t)\right|^{p_1}\,dt\right]\\
&\le \mathbb{E}\int_0^T \left\{J\left(1+\left|\mathbb{E}^{\mathscr{F}_t}\left[Y^n_{\varphi(n,t)}\right]\right|^l + \left|Y^n_t\right|^l\right)\left|\mathbb{E}^{\mathscr{F}_t}\left[Y^n_{\varphi(n,t)} - Y^n_t\right]\right|\right\}^{p_1}\,dt\\
&\le C\mathbb{E}\left[\int_0^T\left(1+\left|\mathbb{E}^{\mathscr{F}_t}\left[Y^n_{\varphi(n,t)}\right]\right|^{lp_1} + |Y^n_t|^{lp_1}\right)\right.\\
&\qquad \times \left.\left|\mathbb{E}^{\mathscr{F}_t}\left[\int_t^{\varphi(n,t)} f^n(s,\mathbb{E}^{\mathscr{F}_s}[Y^n_{\varphi(n,s)}],Z^n_s)\,ds\right]\right|^{p_1} dt\right]\\
&\le C\mathbb{E}\left[\int_0^T\left(1+|\mathbb{E}^{\mathscr{F}_t}[Y^n_{\varphi(n,t)}]|^{lp_1} + |Y^n_t|^{lp_1}\right)\right.\\
&\qquad \times \left.\left(\mathbb{E}^{\mathscr{F}_t}\left[\int_t^{\varphi(n,t)}\left[n^\alpha\left(1+|Y^n_{\varphi(n,t)}|\right) + |Z^n_s|\right]ds\right]\right)^{p_1} dt\right]\\
&\le Cn^{(\alpha-1)p_1}\int_0^T\left(1+\mathbb{E}\left[\left|Y^n_{\varphi(n,t)}\right|^{p_1(l+1)}\right] + \mathbb{E}\left[\left|Y^n_t\right|^{p_1(l+1)}\right]\right)dt\\
&\quad + C\mathbb{E}\left[\int_0^T\left(\frac{T}{n}\mathbb{E}^{\mathscr{F}_t}\left[\int_t^T |Z^n_s|^2 ds\right]\right)^{\frac{p_1(l+1)}{2}} dt\right]\\
&\le Cn^{(\alpha-1)p_1} + Cn^{-\frac{(l+1)p_1}{2}} \le Cn^{(\alpha-1)p_1} + Cn^{-\frac{p_1}{2}},
\end{aligned}
$$

in which (4.9) is used. By Lemma 4.4, we get that for any $\gamma \in (0,1)$,

$$
\mathbb{E}\left[\sup_{0\le t\le T}\phi(t)\left|Y^n_t - Y_t\right|^{p_1\gamma}\right] \le C(n^{-\alpha p_1} + n^{(\alpha-1)p_1} + n^{-\frac{p_1}{2}})^\gamma. \quad (4.12)
$$

In view of (4.8), the parameter α is optimally chosen to be $\frac{1}{2}$. Then, for any $p \in [1, \frac{p_0}{2l+1})$, we have

$$
\left(\mathbb{E}\left[\sup_{0\le t\le T}\left|Y^n_t - Y_t\right|^p\right]\right)^{\frac{1}{p}} \le \frac{C}{\sqrt{n}}.
$$

Step 3: We prove that $\{Z^n\}_{n\ge1}$ converges to Z in $\mathcal{H}^p_T(\mathbb{R}^k)$.

Taking $p_1 = 2$ in (4.11), we have

$$
\begin{aligned}
&\frac{e^{-CT}}{2}\int_0^T |Z^n_t - Z_t|^2\,dt\\
&\quad \le -2\int_0^T \phi(t)\left\langle Y^n_t - Y_t, (Z^n_t - Z_t)dW_t\right\rangle\\
&\qquad + C\int_0^T \left|f^n(t,\mathbb{E}^{\mathscr{F}_t}\left[Y^n_{\varphi(n,t)}\right],Z_t) - f(t,\mathbb{E}^{\mathscr{F}_t}\left[Y^n_{\varphi(n,t)}\right],Z_t)\right|^2 dt\\
&\qquad + C\int_0^T \left|f(t,\mathbb{E}^{\mathscr{F}_t}\left[Y^n_{\varphi(n,t)}\right],Z_t) - f(t,Y^n_t,Z_t)\right|^2 dt.
\end{aligned}
$$

If $p \in [2, \frac{p_0}{2l+1})$, we get

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T |Z^n_t - Z_t|^2\,dt\right)^{\frac{p}{2}}\right]\\
&\quad \le C\mathbb{E}\left[\left|\int_0^T \phi(t)\left\langle Y^n_t - Y_t, (Z^n_t - Z_t)dW_t\right\rangle\right|^{\frac{p}{2}}\right]\\
&\qquad + C\mathbb{E}\left[\int_0^T \left|f^n(t,\mathbb{E}^{\mathscr{F}_t}[Y^n_{\varphi(n,t)}],Z_t) - f(t,\mathbb{E}^{\mathscr{F}_t}[Y^n_{\varphi(n,t)}],Z_t)\right|^p dt\right]\\
&\qquad + C\mathbb{E}\left[\int_0^T \left|f(t,\mathbb{E}^{\mathscr{F}_t}\left[Y^n_{\varphi(n,t)}\right],Z_t) - f(t,Y^n_t,Z_t)\right|^p\,dt\right]\\
&\quad \le C\mathbb{E}\left[\left(\int_0^T |Y^n_t - Y_t|^2|Z^n_t - Z_t|^2 dt\right)^{\frac{p}{4}}\right] + C[M_1(n,p) + M_2(n,p)]\\
&\quad \le \mathbb{E}\left[\frac{C^2}{2}\sup_{0\le t\le T}|Y^n_t - Y_t|^p + \frac{1}{2}\left(\int_0^T |Z^n_t - Z_t|^2 dt\right)^{\frac{p}{2}}\right] + Cn^{-\frac{p}{2}},
\end{aligned}
$$

which yields that

$$
\left(\mathbb{E}\left[\left(\int_0^T |Z^n_t - Z_t|^2\,dt\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}} \le \frac{C}{\sqrt{n}}.
$$

If $p \in [1,2)$, we can use Jensen's inequality to get

$$
\mathbb{E}\left[\left(\int_0^T |Z^n_t - Z_t|^2\,dt\right)^{\frac{p}{2}}\right] \le \left(\mathbb{E}\left[\int_0^T |Z^n_t - Z_t|^2\,dt\right]\right)^{\frac{p}{2}} \le \left(\frac{C}{\sqrt{n}}\right)^p.
$$

This completes the proof. □

CRediT authorship contribution statement

Yunzhang Li: Methodology, Investigation, Writing - original draft. **Shanjian Tang:** Conceptualization, Investigation, Supervision, Validation, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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