Potential Analysis of multiplicative functionals and Feynman-Kac formula

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Contents

Chapter 1

Introduction

This monograph covers the systematic works involving multiplicative functionals of Markov processes and its related probabilistic potential theory. For an introduction of Markov processes and potential theory, we refer readers to [3] and [16]. For symmetric Markov processes and Dirichlet forms, we refer readers to [**?**].

If a Markov process is absolutely continuous relative to another, then the density is a super-martingale multiplicative functional. Absolute continuity is one of the most important transformations in theory of Markov processes. A general super-martingale multiplicative functional may be written into a martingale multiplicative functional and a decreasing multiplicative functional. We shall first investigate the potential properties of decreasing multiplicative functionals in general framework of Markov processes, as we define the Revuz measures and prove various identities, which could be called Feynman-Kac formula in general, concerning Revuz measures, energy functionals, excessive measures and excessive functions. These formulae generalize various potential identities which San Diego based probabilists, including P.J. Fitzsimmons, R.K.Getoor, M.J. Sharpe, J. Steffens, J. Mitro, formulated in 1980's and 1990's. Using those general results, we then characterize the killing transform of symmetric Markov processes as the subordination in theory of Dirichlet forms, and generalize the classical Feynman-Kac formula.

The martingale part of multiplicative functional is more delicate. In the last chapter, we formulate the Dirichlet form of a symmetric Markov process which is absolutely continuous with respect to a given symmetric Markov process with a martingale density.

1.1 Basic notion of Markov processes

The notation for the objects associated with X basically follows the standard reference books [3], [17], [37], [16], which should be browsed before reading this manuscript. We assume that

$$X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, \mathsf{P}^x)$$

is a right Markov process as defined in §8 of [37] with state space $(E, \mathscr{E})$, semigroup (P_t) and resolvent (U^q). To be more precise, E is a separable Radon space and $\mathscr{E}$ is the Borel σ-algebra of E. Its precise definition is quite complicated to state. However we could say it is almost a strong Markov process with right continuous sample paths. A cemetery point Δ is adjoined to E as an isolated point and $E_\Delta := E \cup \{\Delta\}$, $\mathscr{E}_\Delta := \sigma(\mathscr{E} \cup \{\{\Delta\}\}$. Let $\zeta := \inf\{t : X_t = \Delta\}$, the life time of X. Any sample path will stay at Δ forever after ζ. The filtration $(\mathscr{F}_t)$ is the augmented natural filtration of X and satisfies the usual condition.

A universally measurable function $f \geq 0$ is α-excessive if $e^{-\alpha t} P_t f \uparrow f$ as $t \downarrow 0$. Another important σ-algebra on E is $\mathscr{E}^e$ which is generated by all α-excessive functionals with $\alpha > 0$. Similarly a σ-finite measure ξ is α-excessive if $e^{-\alpha t} \xi P_t \uparrow \xi$. We use S^α and Exc^α to denote the set of α-excessive functions and measures respectively. (The superscript will always be dropped when it is zero.) An excessive function or measure of the form $U^\alpha g$ or μU^α is called an α-potential. A dissipative measure is an increasing limit of a sequence of potentials and any measure in Exc^α with $\alpha > 0$ is dissipative.

The energy functional L (of X) is defined on $\mathrm{Exc} \times \mathscr{S}$ by

$$L(\xi, f) = \sup\{\mu(f) : \mu U \leq \xi\}. \tag{1.1.1}$$

Refer to [Ge] for the basic properties of L, among which are $L(\mu U, u) = \mu(u)$ and $L(m, Uf) = m_d(f)$ where m_d is the dissipative part of m. The energy functional is continuous with respect to increasing limit of excessive measures and functions. The capacity of a Borel subset B with respect to $m \in \mathrm{Exc}$ may be defined as

$$\Gamma(B) = L(m, P_B 1), \tag{1.1.2}$$

where $P_B 1 = \mathsf{P}^{\cdot}(T_B < \infty)$.

Multiplicative functionals of Markov processes and related transformations were systematically investigated and described in [3] in 1968 and later in [37]. These are among the most important constructions in the theory of Markov processes

It is always interesting and important to explore various kinds of relationships between X and its M-subprocesses. A great deal of work has been done in this subject. In particular, due to the powerful Kuznetsov measure, a number of significant results have become known. But many still remain uncovered. The first part of this manuscript treats transformations of Markov processes by their multiplicative functionals.

Definition 1.1.1 A real-valued process $M = (M_t : t \geq 0)$ is called a multiplicative functional (MF) of X if (1) $t \mapsto M_t(\omega)$ is decreasing, right continuous and has values in $[0, 1]$ for each $\omega \in \Omega$; (2) M is adapted, i.e., $M_t \in \mathscr{F}_t$ for any $t \geq 0$; (3) $M_{t+s}(\omega) = M_t(\omega) \cdot M_s(\theta_t \omega)$ for any $s, t \geq 0$ and $\omega \in \Omega$. In addition, an MF M is exact provided for any $t > 0$ and every sequence $t_n \downarrow\downarrow 0$,

$$M_{t-t_n} \circ \theta_{t_n} \to M_t \quad a.s. \ as \ n \to \infty.$$

If (1) is replaced by a weaker condition: $\mathsf{E}^x M_t \leq 1$ for all $t > 0$, M is called a super-martingale multiplicative functional since M is a super-martingale in this case. We shall focus on decreasing MF's exclusively in first a few chapters and turn to super-martingale MF's in Chapter 7.

Let MF(X) (or MF if no confusion will be caused) be the set of all exact multiplicative functionals of X. For any MF M write $S_M := \inf\{t \geq 0 : M_t = 0\}$, $E_M := \{x \in E : \mathsf{P}^x(M_0 = 1) = 1\}$, for the lifetime and the set of permanent points of M, respectively. Let $R := \inf\{t > 0 : X_t \in E_M^c\}$ and $R' := \inf\{t \geq 0 : X_t \in E_M^c\}$. Clearly E_M is also the set of irregular points of S_M, i.e., $E_M = \{x \in E : \mathsf{P}^x(S_M > 0) = 1\}$. Let $\mathrm{MF}_+ := \{M \in \mathrm{MF} : S_M > 0 \text{ a.s.}\}$ and $\mathrm{MF}_{++} := \{M \in \mathrm{MF} : M$ does not vanish, i.e., $S_M \geq \zeta$ a.s.$\}$. If E_M is optional, M is called a right MF. If M is exact, E_M is finely open and consequently M is right.

We define for each $x \in E_M$ a probability measure on Q^x on $(\Omega, \mathscr{F}^0)$ by

$$Q^x(Z) := \mathsf{P}^x \int_{]0,\infty]} Z \circ k_t d(-M_t), \quad Z \in b\mathscr{F}^0 \tag{1.1.3}$$

where $M_\infty := 0$ and $(k_t)_{t \geq 0}$ are the killing operators on Ω defined by $k_t\omega(s) = \omega(s)$ if $t > s$ and $k_t\omega(s) = \Delta$ if $t \leq s$. If M is right, then $(\Omega,\ X_t,\ Q^x)$ is also a right Markov process, called the M-subprocess of X and denoted X^M or (X, M), with state space $(E_M, \mathscr{E}_M)$. Clearly the semigroup (Q_t) and resolvent (V^q) of X^M are given by

$$\begin{aligned} Q_t f(x) &= \mathsf{P}^x[f(X_t)M_t]; \\ V^q f(x) &= \mathsf{P}^x \int_0^\infty e^{-qt} f(X_t) M_t \, dt, \end{aligned} \tag{1.1.4}$$

and $Q_t(x, \cdot) = V^q(x, \cdot) = 0$ for any $x \notin E_M$. The set of q-excessive functions (resp. excessive measures) of X^M is denoted by $\mathscr{S}^q(M)$ (resp. $\mathrm{Exc}^q(M)$).

An $(\mathscr{F}_t)$-stopping time $T : \Omega \to R_+$ is called a terminal time if $T = t + T \circ \theta_t$ identically on $\{t < T\}$. If T is a terminal time, $1_{[0,T[}(t)$ is an MF of X. Write $\mathrm{Exc}^q(T) := \mathrm{Exc}^q(1_{[0,T[})$ and $\mathscr{S}^q(T) := \mathscr{S}^q(1_{[0,T[})$. It is easy to see that S_M is a terminal time if $M \in \mathrm{MF}(X)$.

Definition 1.1.2 Let $M \in \mathrm{MF}$. A positive, increasing, right continuous process $A = (A_t : t \geq 0)$ is a raw M-additive functional (of X) provided $A_t < \infty$ for $t < S \wedge \zeta$ and $A_{s+t} = A_t + M_t \cdot A_s \circ \theta_t$ a.s. for each t and s. A raw M-additive functional A is an M-additive functional (of X) if A is adapted.

Let $\mathrm{RAF}(M)$ and $\mathrm{AF}(M)$ denote the sets of raw M-additive functionals and M-additive functionals, respectively. Write RAF for $\mathrm{RAF}(1)$, the set of raw additive functionals, and AF for $\mathrm{AF}(1)$, the set of additive functionals of X. For a terminal time T, $\mathrm{RAF}(T) := \mathrm{RAF}(1_{[0,T[})$ and $\mathrm{AF}(T) := \mathrm{AF}(1_{[0,T[})$.

Additive functionals are certainly among the most important concepts in probabilistic potential theory. Assume that A is an additive functional. We may associate a measure, called Revuz measure of A with respect to $m \in \mathrm{Exc}$, as in [34] and [35]

$$\rho_A^m(f) := \lim_{t\downarrow 0} \frac{1}{t} \mathsf{P}^m \int_0^t f(X_s) dA_s. \tag{1.1.5}$$

The right side is actually increasing as t decreases, see, e.g., [37]. Let

$$U_A^\alpha f(x) := \mathsf{P}^x \int_0^\infty e^{-\alpha t} f(X_t) dA_t$$

the α-potential of A. Then

$$\rho_A^m(f) = \lim_{\alpha\to\infty} \alpha \langle m, U_A^\alpha f\rangle$$

and more important

$$\rho_A^m(f) = L(m, U_A f), \tag{1.1.6}$$

when m is dissipative.

A PCAF means a positive continuous additive functional of X. A PCAF may be used to define time change, which is another important transformation in the theory of Markov processes and will be treated in later chapters. Given a PCAF A of X. Define

$$\begin{aligned} \tau_t &:= \inf\{s > 0 : A_s > t\}, \\ R_A &:= \inf\{t > 0 : A_t > 0\} \\ F &:= \{x : \mathsf{P}^x(R_A = 0) = 1\} \\ T &:= \inf\{t > 0 : X_t \in F\}, \end{aligned}$$

where (τ_t) is the right continuous inverse of A and F is the fine support of A. It is known that $\tau_0 = R_A = T$ (see §64 [37]). Let $\Phi(x) = \mathsf{E}^x(e^{-T})$. Then $F = \{\Phi = 1\} \in \mathscr{E}^e$, where $\mathscr{E}^e$ is the σ-algebra generated by 1-excessive functions of X. Obviously $\{\tau_t : t \geq 0\}$ is an increasing family of stopping times and $\tau_0 = R_A$. Set $\tilde{X}_t = X_{\tau_t}$, $\tilde{\mathscr{F}}_t = \mathscr{F}_{\tau_t}$, and $\tilde{\theta}_t = \theta_{\tau_t}$. Then the process

$$\tilde{X} = (\Omega, \mathscr{F}, \tilde{\mathscr{F}}_t, \tilde{X}_t, \tilde{\theta}_t, \mathsf{P}^x)$$

is a right Markov process on space F with life $\tilde{\zeta} = A_\infty$. This process is called the time change of X by A.

We now introduce the useful Kuznetsov measures. Let W be the space of path $w: \mathbf{R} \longrightarrow E \cup \{\Delta\}$ that are E-valued and right continuous on an open interval $]\alpha(w), \beta(w)[$ and take the value Δ elsewhere. We denote by $[\Delta]$ the path which constantly equals to Δ. Let $Y = (Y_t : t \in \mathbf{R})$ denote the coordinate process on W, $Y_t(w) = w(t)$. The shifts $\sigma_t : W \longrightarrow W$ are defined by $Y_s \circ \sigma_t = Y_{s+t}$. Put $\mathscr{G}_t^0 = \sigma\{Y_s : s \le t\}$ and $\mathscr{G}^0 = \mathscr{G}_\infty^0$. Then for any $m \in \text{Exc}$, there exists a unique σ-finite measure Q_m on $(W, \mathscr{G}^0)$ not charging $\{[\Delta]\}$ such that if $t_1 < t_2 < \cdots < t_n$,

$$\begin{aligned} &Q_m(\alpha < t_1, Y_{t_1} \in dx_1, \cdots Y_{t_n} \in dx_n,\, t_n < \beta) \\ &\quad = m(dx_1) P_{t_2 - t_1}(x_1,\, dx_2) \cdots P_{t_n - t_{n-1}}(x_{n-1},\, dx_n). \end{aligned}$$

The measure Q_m is called the Kuznetsov measure of (X and) m. Clearly and importantly it is translation-invariant; that is, $\sigma_t(Q_m) = Q_m$ for each $t \in \mathbf{R}$.

Chapter 2

Revuz measures and related formulas

In this chapter we shall discuss potential theory of multiplicative functionals. In §2.1, we will define Revuz measure for multiplicative functionals. We shall see that this actually generalizes original definition of Revuz measure for additive functionals since when $M \in \mathrm{MF}$ never vanishes, the Revuz measure of M is identical to that of its logarithm, an AF. In §2.2 and §2.3 we shall study the relationship of energy functionals and capacity related to X and its subprocess induced by a multiplicative functional, respectively.

2.1 Revuz Measures of Multiplicative Functionals

Revuz measures were first introduced for ordinary additive functionals in [34]. Let $m \in \mathrm{Exc}$ and $A \in \mathrm{RAF}$. Then the Revuz measure of A relative to m is defined by

$$\rho_A^m(f) := \uparrow \lim_{t \downarrow 0} \frac{1}{t} \mathsf{P}^m \int_0^t f(X_s)\, dA_s. \qquad (2.1.1)$$

The measure ρ_A^m charges no m-polar sets. If A is continuous, ρ_A^m charges no semipolar sets and uniquely determines A.

Now let $M \in$ MF; the right side of (2.1.1) makes perfect sense due to (2.5) of [13] for $A \in \mathrm{RAF}(M)$ and $m \in \mathrm{Exc}(M)$. We use the same notation for this measure and also call it the Revuz measure of A relative to m. It is easy to see that the process $(1 - M_t)$ is an M-additive functional. We denote the corresponding Revuz measure by ρ_M^m for $m \in \mathrm{Exc}(M)$ and call it the Revuz measure of M.

We know that if $A \in$ RAF and $m \in \mathrm{Dis}^q$, we have the following representation

$$ {}^q\rho_A^m(f) = L^q(m, U_A^q f), \tag{2.1.2} $$

where ${}^q\rho_A^m$ is the Revuz measure of A in the context of q-subprocess and $\rho_A^m = {}^q\rho_A^m$ by (8.10) of [16]. We are going to give a similar result for Revuz measures of M-additive functionals. Let U_A^q be the q-potential operator of $A \in \mathrm{RAF}(M)$; that is, $U_A^q f(x) := \mathsf{P}^x \int_0^\infty e^{-qt} f(X_t)\, dA_t$, and U_M^q the q-potential operator of $(1 - M_t)$.

Proposition 2.1.1 Let $M \in$ MF and $A \in \mathrm{RAF}(M)$. Then for $q \geq 0$,

(a) $U_A^q f \in \mathscr{S}^q(M)$;

(b) ${}^q\rho_A^\xi = \mu U_A^q$ if $\xi = \mu V^q$;

(c) ${}^q\rho_A^m(f) = L^{q,M}(m, U_A^q f)$ for any $f \in p\mathscr{E}$ if $m \in \mathrm{Dis}^q(M)$, where $L^{q,M}$ is the energy functional of the $(e^{-qt}M_t)$-subprocess of X;

(d) if there exists $\{\mu_n\}$ such that $\mu_n V^q \uparrow m$, then $\mu_n U_A^q \uparrow {}^q\rho_A^m$.

Proof. Part (a) is trivial. For part (b), using Fubini Theorem, we find

$$ \begin{aligned} \mathsf{P}^{\mu V^q}(A_t) &= \mathsf{P}^\mu \int_0^\infty e^{-qs} \mathsf{P}^{X_s}(A_t) \cdot M_s ds = \mathsf{P}^\mu \int_0^\infty e^{-qs} A_t \circ \theta_s \cdot M_s ds \\ &= \mathsf{P}^\mu \int_0^\infty e^{-qs} \int_s^{t+s} dA_r ds = \mathsf{P}^\mu \int_0^\infty dA_r \int_{(r-t)^+}^{r} e^{-qs} ds. \end{aligned} $$

Thus we have

$$\frac{1}{t}\mathsf{P}^{\mu V^q}(A_t) = \mathsf{P}^\mu \int_0^\infty dA_r \frac{1}{t}\int_{(r-t)^+}^r e^{-qs}ds \uparrow \mu U_A^q 1$$

as $t \downarrow 0$. Replacing A by $f * A$ for $f \in p\mathscr{E}$ gives (b).

(c) Since the proof is the same for all $q \geq 0$, we shall write it only for $q = 0$. By the fact that $m \in \mathrm{Dis}(M)$, there exist $\{\mu_n\}$ such that $\mu_n V \uparrow m$ and $\{f_k\}$ such that $Vf_k \uparrow U_A f$ a.e. m and a.e. μ_n for each n due to (2.14) of [16]. Then

$$\begin{aligned}\rho_A^m(f) &=\uparrow \lim_n \rho_A^{\mu_n V}(f) =\uparrow \lim_n \mu_n U_A(f)\\ &=\uparrow \lim_n \uparrow \lim_k \mu_n(Vf_k) =\uparrow \lim_k L^M(m, Vf_k)\\ &= L^M(m, U_A f).\end{aligned}$$

Part (d) follows from (c) directly. □

For $A \in \mathrm{RAF}$ and $M \in \mathrm{MF}$, define a process $(M_- * A)_t := \int_0^t M_{s-}\, dA_s$. It is easy to check that $M_- * A \in \mathrm{RAF}(M)$.

Theorem 2.1.2 Let $m \in \mathrm{Exc}$, $A \in \mathrm{RAF}$ and $M \in \mathrm{MF}_{++}$. Then $\rho_{M_- * A}^m = \rho_A^m$.

Proof. The Stieltjes logarithm of M is defined by

(2.1.3) $$(\mathrm{slog} M)_t := \int_0^t 1_{\{s<S_M\}} \frac{d(-M_s)}{M_{s-}}.$$

Clearly $\mathrm{slog}M \in \mathrm{AF}(S_M)$. We write $\mathrm{slog}M$ as $[M]$ when M never vanishes. It is easy to see that $[M] \in \mathrm{AF}$ and $\Delta[M]_t = [M]_t - [M]_{t-} < 1$ for $t \geq 0$. (Conversely if $A \in \mathrm{AF}$ and $\Delta A_t < 1$ for each $t \geq 0$, there exists uniquely $M \in \mathrm{MF}$, the Stieltjes exponential of A, such that $[M] = A$.)

Now for $f \in p\mathscr{E}$, $x \in E$ and $q \geq 0$ with $U_A^q f(x) < \infty$,

$$\begin{aligned}U_{[M]}^q U_{M_- * A}^q f(x) &= \mathsf{P}^x \int_0^\infty e^{-qt}\frac{d(-M_t)}{M_{t-}} P^{X_t} \int_0^\infty e^{-qs} f(X_s) M_{s-}\, dA_s\\ &= \mathsf{P}^x \int_0^\infty \frac{d(-M_t)}{M_{t-}M_t} \int_t^\infty e^{-qs} f(X_s) M_{s-}\, dA_s\end{aligned}$$

$$
\begin{aligned}
&= \mathsf{P}^x \int_0^\infty d\frac{1}{M_t} \int_t^\infty e^{-qs} f(X_s) M_{s-}\, dA_s \\
&= -\mathsf{P}^x \int_0^\infty e^{-qs} f(X_s) M_{s-}\, dA_s \\
&\quad + \mathsf{P}^x \int_0^\infty \frac{1}{M_{t-}} e^{-qt} f(X_t) M_{t-}\, dA_s \\
&= U_A^q f(x) - U_{M_-*A}^q f(x).
\end{aligned}
$$

Thus we have the identity

$$U_A^q = U_{M_-*A}^q + U_{[M]}^q U_{M_-*A}^q. \tag{2.1.4}$$

Since $m \in \mathrm{Dis}^q = \mathrm{Exc}^q$ for any $q > 0$, there exists $\{\mu_n\}$ such that $\mu_n U^q \uparrow m$. Let $\nu_n = \mu_n + \mu_n U_{[M]}^q$. Then using a well-known identity

$$U^q = V^q + U_{[M]}^q V^q, \tag{2.1.5}$$

we have $\nu_n V^q = \mu_n(V^q + U_{[M]}^q V^q) = \mu_n U^q \uparrow m$. Thus by (2.1.4)

$$ {}^q\rho_A^m = \uparrow \lim_n \mu_n U_A^q = \uparrow \lim_n \nu_n U_{M_-*A}^q = {}^q\rho_{M_-*A}^m.$$

Finally $\rho_A^m = {}^q\rho_A^m = {}^q\rho_{M_-*A}^m = \rho_{M_-*A}^m$. □

Remark. By the last sentence in the proof, we actually need only to prove $\rho_A^m = \rho_{M_-*A}^m$ for the case $m \in \mathrm{Dis}$. We will use this technique if applicable without explanation.

Corollary 2.1.1 Let $M \in \mathrm{MF}_{++}$ and $m \in \mathrm{Exc}$. Then $\rho_M^m = \rho_{[M]}^m$; that is, the Revuz measure of M is the same as that of the Stieltjes logarithm of M.

Proof. Since M never vanishes, $[M] \in \mathrm{AF}$ and $M_- * [M]_t = 1 - M_t$. By the theorem above $\rho_{[M]}^m = \rho_{M_-*[M]}^m = \rho_M^m$. □

This result hints that the notion of Revuz measures of multiplicative functionals generalizes in some sense that of additive functionals.

Remark. The proof of Proposition 2.1.1 is analogous to that of (8.11) and (8.13) of [16].

2.2 Formulas on Energy Functionals

For $M \in \mathrm{MF}$, define

$$P_M^q f(x) := \begin{cases} U_M^q f(x), & x \in E_M; \\ f(x), & x \in E_M^c. \end{cases}$$

(Clearly $P_M^q = U_M^q$ if $M \in \mathrm{MF}_+$.) We have Dynkin's formula:

$$U^q - V^q = P_M^q U^q. \tag{2.2.1}$$

Here we will give a similar formula for U_M^q. Recall that R' and R are the first entrance time and hitting time of E_M^c as in §1.1. First it is easy to check that

1) $R' \le R$ a.s. Precisely $\mathsf{P}^x(R = R' > 0) = 1$ for $x \in E_M$ (since E_M is finely open), $\mathsf{P}^x(R = R' = 0) = 1$ for $x \in (E_M^c)^r$ and $\mathsf{P}^x(R > R' = 0) = 1$ for $x \in E_M^c - (E_M^c)^r$.

2) If M is exact, $S_M \le R' \le R$ a.s. Precisely, $\mathsf{P}^x(R' = R \ge S_M) = 1$ for $x \in E_M$ and $\mathsf{P}^x(R' = S_M = 0) = 1$ for $x \in E_M^c$.

Lemma 2.2.1 If $M \in \mathrm{MF}$, then

$$V_{R'}^q - V^q = U_M^q V_{R'}^q,$$

where $(V_{R'}^q)$ is the resolvent of (X, R').

Proof. It is easy to see the both sides vanish for $x \in E_M^c$. Hence we need only to show Lemma 2.2.1 for $x \in E_M$, in which case $R' = R$ P^x-a.s. From the fact that E_M^c is finely closed, it follows that $X_R = X_{R'} \in E_M^c$ P^x-a.s. and thus $R' \circ \theta_t + t = R'$ P^x-a.s. on $\{t \le R'\}$. Since M is exact, we have $S_M \le R'$ and further

$$\begin{aligned} U_M^q V_{R'}^q f(x) &= \mathsf{P}^x \int_0^\infty \int_0^\infty e^{-q(s+t)} f(X_{s+t}) 1_{\{s < R' \circ \theta_t\}} ds\, 1_{\{t \le S_M\}} d(-M_t) \\ &= \mathsf{P}^x \int_0^\infty \int_t^\infty e^{-qs} f(X_s) 1_{\{s < R'\}} ds\, d(-M_t). \end{aligned}$$

Then changing the order of integration the lemma follows. □

Theorem 2.2.1 Let $M \in$ MF. If $m \in$Exc(R') and $u \in \mathscr{S}(R')$, then

$$L^M(m,u) = L^{R'}(m,u) + \rho_M^m(u). \tag{2.2.2}$$

Proof. Since M is exact, E_M is nearly optional and finely open. Hence S_M and R' are right terminal times. Then $X^{R'}$ is a right process and Exc$(R') \subset$Exc(M). Therefore (2.2.2) makes sense.

First we will show that Dis$(R') \subset$Dis(M). In fact, for $m \in$Dis(R'), there exists $g > 0$ with $m(g) < \infty$ such that $m\{V_{R'}g = \infty\} = 0$ where $(V_{R'}^q)$ is the resolvent of $X^{R'}$. Since $V \leq V_{R'}$, $m\{Vg = \infty\} \leq m\{V_{R'}g = \infty\} = 0$, i.e., $m \in$Dis(M).

Next for $g \in p\mathscr{E}$ and $m \in$Dis(R'), we have

$$L^M(m, Vg) = m(g) = L^{R'}(m, V_{R'}g).$$

Then using Lemma 2.2.1, it follows that

$$\begin{aligned} L^M(m, V_{R'}g) &= L^{R'}(m, V_{R'}g) + L^M(m, U_M V_{R'}g) \\ &= L^{R'}(m, V_{R'}g) + \rho_M^m(V_{R'}g). \end{aligned}$$

Now for $u \in \mathscr{S}(R')$, by (2.14) of [16], there exists a increasing sequence of potentials $(V_{R'}g_n)$ such that $V_{R'}g_n \uparrow u$ a.e. m. Let $h = \lim_n V_{R'}g_n$. Then $h \in \mathscr{S}(R')$ and $m\{h < u\} = 0$. Since $\{h < u\}$ is finely open, $\{h < u\}$ is m-polar. Thus $V_{R'}g_n \uparrow u$ a.e. ρ_M^m which does not charge m-polar sets. Therefore by (3.4ii) of [16], (2.2.2) holds for $m \in$Dis(R').

Finally we need only to show (2.2.2) for $m \in$Con(R'). We have two cases.

1) If $m \in$Con$(R')\cap$Inv(M), then by Lemma 2.2.1 for $q > 0$,

$$m = qmV_{R'}^q = qmV^q + qmU_M^q V_{R'}^q = m + qmU_M^q V_{R'}^q.$$

Hence $mU_M^q V_{R'}^q = 0$ and $mV^q = mV_{R'}^q$ even when $q = 0$. Since $V_{R'}^q \geq V^q$, the last equality yields $V_{R'}^q(x, \cdot) = V^q(x, \cdot)$ for m a.e. $x \in E_M$. But $m \in$Con(R') so if $u > 0$ then $Vf = V_{R'}f = \infty$ a.e. m (since m is carried by E_M), hence $m \in$Con(M). Since $V_{R'}^q 1 > 0$ everywhere on E_M, $mU_M^q V_{R'}^q = 0$ implies that $mU_M^q = 0$. Hence $\rho_M^m = 0$, i.e., all items in (2.2.2) vanish.

2) Now assume that $m \in$Con$(R')\cap$Pur(M) now. By (3.6i) of [16], we have

$$\begin{aligned} L^M(m,u) &= \uparrow \lim_{q\to\infty} q\langle m - qmV^q, u\rangle \\ &= \uparrow \lim_{q\to\infty} q\langle qmU^q_M V^q_{R'},\, u\rangle \\ &= \uparrow \lim_{q\to\infty} \langle m,\, qU^q_M qV^q_{R'} u\rangle. \end{aligned}$$

The second equality holds since m is invariant for (X, R'). But $qV^q_{R'}u \uparrow u$ and $\langle m, qU^q_M g\rangle \uparrow \rho^m_M(g)$ for any $g \in p\mathscr{E}$. By an elementary fact, we have

$$\uparrow \lim_{q\to\infty} \langle m,\, qU^q_M qV^q_{R'} u\rangle = \lim_{q\to\infty} \lim_{p\to\infty} \langle m,\, qU^q_M pV^p_{R'} u\rangle = \rho^m_M(u).$$

Thus $L^M(m,u) = \rho^m_M(u)$ which is just (2.2.2) in this case. □

Remark. If $E_M = E$ or equivalently $S_M > 0$ a.s., then (2.2.2) becomes

$$L^M(m,u) = L(m,u) + \rho^m_M(u) \tag{2.2.3}$$

for any $m \in$ Exc and $u \in \mathscr{S}(X)$. Furthermore if M never vanishes, then by Corollary 2.1.1, we have

$$L^M(m,u) = L(m,u) + \rho^m_{[M]}(u), \tag{2.2.4}$$

which is (1.1) of [12].

Now we are going to use the Kuznetsov measure to give some more interesting formulas on energy functionals. Let $m \in$ Exc and Q_m be the Kuznetsov measure of m. Then by (4.6) of [12] the balayage operator is given by

$$R_M m(f) = Q_m[f\circ Y_0 \cdot (1 - N_0)], \tag{2.2.5}$$

where $N = (N_t)$ is the extension of M on W, i.e., if we define for $\alpha < s \le t$,

$$N(s,t) := \begin{cases} M_{t-s}\circ\theta_s, & \alpha < s < t; \\ 1 & \alpha < s = t, \end{cases}$$

where (θ_t) are truncated shift operators on W: $\theta_t w(s) = w(t+s)$ if $s > 0$, $\theta_t w(s) = \Delta$ if $s \le 0$, then $N_t :=\uparrow \lim_{s\downarrow\alpha} N(s,t)$.

Proposition 2.2.2 (1) Let S be a stationary stopping time on W and define $\mu_S(f) := Q_m(f\circ Y_S;\ 0 < S < 1)$. Then

$$\mu_S V(f) = Q_m[f\circ Y_0 \cdot N(S,0); \alpha < S < \beta; S < 0]; \tag{2.2.6}$$

(2) if $m \in \mathrm{Dis}$, then there exists a sequence of stationary times $\{S_n\}$ such that $S_n \downarrow\downarrow \alpha$ a.s. Q_m and $\mu_{S_n} V \uparrow m - R_M m$.

Proof. (1) By the direct computation,

$$\begin{aligned}
\mu_S V(f) &= \int_0^\infty \mu_S Q_t(f)dt \\
&= \int_0^\infty Q_m[f(Y_{t+S}) \cdot N(S, t+S) \cdot 1_{\{0<S<1;\alpha<S<\beta\}}]dt \\
&= \int_{-\infty}^\infty Q_m[f\circ Y_t \cdot N(S,t) \cdot 1_{\{S<t;0<S<1;\alpha<S<\beta\}}]dt \\
&= \int_{-\infty}^\infty Q_m[f\circ Y_0 \cdot N(S,0) \cdot 1_{\{S<0;\alpha<S<\beta;-t<S<1-t\}}]\circ\sigma_t\, dt \\
&= Q_m[f\circ Y_0 \cdot N(S,0); S<0; \alpha<S<\beta].
\end{aligned}$$

The fourth equality holds since

$$N(S,0)\circ\sigma_t = M_{-S\circ\sigma_t}\circ\theta_{S\circ\sigma_t+t} = N(S,t).$$

(2) The existence of $\{S_n\}$ has been proved in (6.24) of [16]. Using (1), we have

$$\begin{aligned}
\mu_{S_n} V(f) &= Q_m[f\circ Y_0 \cdot N(S_n,0); S_n < 0; \alpha < S_n < \beta] \\
&\uparrow Q_m(f\circ Y_0 \cdot N_0; \alpha < 0).
\end{aligned}$$

Then by (2.2.5), $\mu_{S_n} V \uparrow m - R_M m$. $\square$

Now we are going to prove the following useful formulas.

Theorem 2.2.3 Let $M \in \mathrm{MF}_+$, $m \in \mathrm{Exc}$ and $u \in \mathscr{S}$. Then

$$L(m,u) = L^M(m - R_M m, u). \tag{2.2.7}$$

Furthermore if $m \in \mathrm{Dis}$, we have

$$L(m,u) = L^M(m, u - P_M u). \tag{2.2.8}$$

Proof. For $m \in \mathrm{Dis}$, (2.2.7) follows directly from Proposition 2.2.2. If $m \in \mathrm{Con}$, then $m - R_M m \in \mathrm{Con}(M)$, which was proved in the proof of (4.2) of [12]. Hence (2.2.7) holds.

Now assume that $m \in \mathrm{Dis}$. By a proof analogous to (4.4) of [9], there exists a sequence $\{T_k\}$ of intrinsic stationary stopping times of $(\mathscr{H}_t^0)$, the reversal filtration of W, i.e., $\mathscr{H}_t^0 := \sigma\{Y_s : s \geq t\}$, such that (a) $T_k \uparrow\uparrow \beta$ a.e. Q_m, (b) $\alpha < T_k < \beta$ if $T_k > -\infty$, (c) $\{T_k \circ b_s = -\infty\} = \{T_k \leq s\}$ and $T_k \circ b_s = T_k$ if $T_k > s$, where (b_s) are the birthing operators on W: $b_s w(r) = w(r)$ if $s < r$; $b_s w(r) = \Delta$ if $s \geq r$. Define $g := 1 - P_M 1 = P^{\cdot}(M_{\zeta-})$ and

$$Q_{m,g}^M(Z) := Q_m \int_{[-\infty,\beta[} Z \circ b_s \, dn_s,$$

for $Z \in b\mathscr{G}^0$, where

$$n_s := \begin{cases} \downarrow \lim_{t \uparrow \beta} N(s,t) = N(t, \beta-), & s \in]\alpha, \beta[; \\ 1, & \beta \leq s; \\ 0, & \alpha \geq s. \end{cases}$$

For any $u \in \mathscr{S}$ the u-transform of X is denoted by $X^{(u)}$ and is a Borel right process on the state space $E_u = \{0 < u < \infty\}$ with semigroup $P_t^{(u)} f = u^{-1} P_t(uf)$. In general the superscript (u) will indicate objects defined relative to u-transform.

By (4.12i) of [47], $Q_{m,g}^M$ is the Kuznetsov measure of gm and $(Q_t^{(g)})$, where $(Q_t^{(g)})$ is the semigroup of the g-transform of (X, M). Now using the properties of T_k

$$\begin{aligned} Q_{m,g}^M(0 < T_k < 1) &= Q_m \int_{[-\infty,\beta[} 1_{]0,1[}(T_k \circ b_s) dn_s \\ &= Q_m \int_{[-\infty,T_k[} 1_{]0,1[}(T_k) dn_s \\ &= Q_m(n_{T_k-}; 0 < T_k < 1). \end{aligned}$$

We can pick a stationary random time S^* such that $Q_m(S^* \notin \mathbf{R}) = 0$. Since n_{T_k-} is (σ_t)-invariant,

$$Q_{m,g}^M(0 < S^* < 1, T_k \in \mathbf{R}) = Q_{m,g}^M(0 < T_k < 1)$$

$$= Q_m(n_{T_k-}; 0 < S^* < 1; T_k \in]\alpha, \beta[).$$

When k tends to infinity, we find

$$\begin{aligned} L^M(m, g) &= (L^M)_g(gm, 1) = Q^M_{m,g}(0 < S^* < 1) \\ &= Q_m(n_{\beta-}; 0 < S^* < 1) \\ &= Q_m(0 < S^* < 1) = L(m, 1), \end{aligned}$$

where $(L^M)_g$ is the energy functional of the g-transform of (X, M). The fourth equality holds due to the fact that $M_0 = 1$ a.s. implies that $n_{\beta-} = N_\alpha = 1$ a.e. Q_m. Finally by (4.16) of [12]

$$\begin{aligned} L(m, u) &= L_u(um, 1) \\ &= (L_u)^M(um, 1 - P^{(u)}_M 1) \\ &= (L^M)_u(um, 1 - P^{(u)}_M 1) \\ &= L^M(m, u - u \cdot P^{(u)}_M 1) \\ &= L^M(m, u - P_M u), \end{aligned}$$

where similarly $(L_u)^M$ is the energy functional of $(X^{(u)}, M)$. The third equality holds since 'killing' transform and h-transform commute. □

The formula (2.2.8) is the same as (2.2.2) when $\rho^m_M(u) = L^M(m, P_M u) < \infty$. As an application of (2.2.8), we have the following significant generalized form of Theorem 2.1.2.

Corollary 2.2.1 Let $M \in \mathrm{MF}_+$, $A \in \mathrm{RAF}$ and $m \in \mathrm{Exc}$. Then we have

$$\rho^m_A = \rho^m_{M_-*A}.$$

Proof. It is easy to check the identity

$$U_A - P_M U_A = U_{M_-*A}. \tag{3.14}$$

Hence by (2.2.8)

$$\rho^m_A(f) = L(m, U_A f) = L^M(m, U_A f - P_M U_A f)$$

$$= L^M(m, U_{M_-*A}f) = \rho^m_{M_-*A}.$$

That completes the proof. □

To conclude this section, we would like to give a direct proof of a very useful formula (2.22) of [13], which states that if $A \in \text{AF}$ is continuous, if $M \in \text{MF}$ and if $m \in \text{Exc}$, then

$$\rho^m_{M*A} = 1_{E_M} \cdot \rho^m_A, \tag{2.2.9}$$

where $(M * A)_t := \int_0^t M_s dA_s$.

Since A is continuous, $M_- * A = M * A$ and it follows from Corollary 2.2.1 that $\rho^m_{M*A} = \rho^m_A$ if $E_M = E$. Hence it suffices to show (2.2.9) when $M = 1_{[0,T[}$ where $T := T_{E^c_M}$. Then by (8.21) of [16]

$$\begin{aligned}\rho^m_{M*A}(f) &= \lim_{t\to 0} \frac{1}{t} \mathsf{P}^m \int_0^t 1_{\{s<T\}} f(X_s) dA_s \\ &= \lim_{t\to 0} \frac{1}{t} Q^{m^*} \int_0^t f(X_s) dA_s \\ &= Q^M_{m^*}\big[\int_{-\infty}^{\infty} \phi(t) f(Y_t) A^*(dt)\big],\end{aligned}$$

where $\phi \in p\mathscr{R}$ with $\int \phi = 1$, $m^* = m|_{E_M}$, A^* the extension of A on W, and $Q^M_{m^*}$ the Kuznetsov measure of m^* and (Q_t). Let J be the closure in $]\alpha, \beta[$ of $\{t : Y_t \in E^c_M\}$, G the set of left endpoints contained in $]\alpha, \beta[$ of the contiguous interval to J, and $G_0 := G \cup \{\alpha\}$. Using (5.11) of [21]

$$\begin{aligned}\rho^m_{M*A}(f) &= Q_m \sum_{s\in G_0} \big[\int_{-\infty}^{\infty} \phi(t) f(Y_t) A^*(dt)\big]\circ b_s \circ k_{d_s} \\ &= Q_m \int_{-\infty}^{\infty} \phi(t) f(Y_t) \sum_{s\in G_0} 1_{]s,d_s[}(t) A^*(dt),\end{aligned}$$

where $d_s = \tau\circ\theta_s + s$ with $\tau := \inf\{t > \alpha : Y_t \in E^c_M\}$. But it is clear that

$$\bigcup_{s\in G_0}]s, d_s[\subset \{t :\ Y_t \in E_M\} \subset \bigcup_{s\in G_0} [s, d_s[.$$

Thus by continuity of A^*, we have

$$\rho^m_{M*A}(f) = Q_m \int_{-\infty}^{\infty} \phi(t) f(Y_t) 1_{E_M}(Y_t) A^*(dt) = \rho^m_A(f \cdot 1_{E_M}).$$

That is exactly (2.2.9).

2.3 A Generalized Formula of Getoor-Steffens

Unless otherwise stated, throughout this section, $M \in \mathrm{MF}$, never vanishes and $m \in \mathrm{Exc}$. Then $[M] \in \mathrm{AF}$. For $B \in \mathscr{E}^e$ and $q \geq 0$, the q-capacity of B, $\Gamma^q(B) = \Gamma^q_m(B)$ is defined by

$$(2.3.1) \qquad \begin{aligned} \Gamma^q(B) &:= \Gamma^q_m(B) := L^q(m, P^q_B 1) = L^q(R^q_B m, 1) \\ \Gamma(B) &:= \Gamma^0(B) \end{aligned}$$

where P^q_B and R^q_B are Hunt's balayage operators. Getoor and Steffens [24] and [25] proved that q-capacity is connected to (0-)capacity by

$$(2.3.2) \qquad \Gamma^q(B) = \Gamma(B) + qR_B m(P^q_B 1)$$

and furthermore if B is finely closed and $(\hat{\mathsf{P}}^x, H)$ is an exit system for B, then

$$(2.3.3) \qquad \Gamma^q(B) = \Gamma(B) + qm(B) + \int_0^\infty (1 - e^{-qt}) \nu_B(dt)$$

where $\nu_B(dt) := \hat{\mathsf{P}}^\rho(T_B \in dt; T_B < \infty)$ and $\rho := \rho^m_H$.

The formula (2.3.3) describes how the capacity of subprocess is related to the original one when the process is killed by an exponential multiplicative functional. In this section, we are going to deduce a similar formula for more general multiplicative functionals.

An exact terminal time T is called strict if $T \circ \theta_T = 0$ a.s. In the case that $T = T_B$, the hitting time of $B \in \mathscr{E}^e$, T is strict if and only if $X_T \in B^r$, where B^r is the set of regular points of B. The set B is called strict if T_B is. Now we will first present two important identities. Let P_T and P^M_T be the balayages in the context of X and its M-subprocess, respectively. Then

$$P^M_T f := Q^{\cdot} f(X_T) = P^{\cdot}(M_T f(X_T)).$$

Lemma 2.3.1 Let T be a strict terminal time. Then the following two identities hold

$$(2.3.4) \qquad P_T + P^M_T U_M P_T = P^M_T + U_M P_T,$$

$$(2.3.5) \qquad P_T + P_T U_{[M]} P^M_T = P^M_T + U_{[M]} P^M_T.$$

Proof. Define for $f \in p\mathscr{E}$,

$$Wf(x) := \mathsf{P}^x \int_{]0,T]} f(X_t)d(-M_t).$$

Since T is strict, we find

$$\begin{aligned} WP_T f(x) &= \mathsf{P}^x \int_{]0,T]} P_T f(X_t)d(-M_t) \\ &= \mathsf{P}^x \int_{]0,T]} f(X_T)\circ\theta_t d(-M_t) \\ &= \mathsf{P}^x f(X_T)(1 - M_T) \\ &= P_T f(x) - P_T^M f(x) \end{aligned}$$

and

$$\begin{aligned} P_T^M U_M f(x) &= \mathsf{P}^x \left(\int_0^\infty f(X_t)d(-M_t) \right) \circ\theta_T M_T \\ &= \mathsf{P}^x \int_0^\infty f(X_{t+T})d(-M_{t+T}) \\ &= \mathsf{P}^x \int_T^\infty f(X_t)d(-M_t) \\ &= U_M f(x) - Wf(x). \end{aligned}$$

Combining these two identities

$$WP_T = P_T - P_T^M ; P_T^M U_M = U_M - W, \tag{2.3.6}$$

the formula (2.3.4) follows.

To prove (2.3.5), define

$$W' f(x) := \mathsf{P}^x \int_{]0,T]} f(X_t)d[M]_t.$$

The similar computation gives

$$W' P_T^M = P_T - P_T^M,\ P_T U_{[M]} = U_{[M]} - W'. \tag{2.3.7}$$

Then the formula (2.3.5) follows easily. □

Remark. 1) If M is continuous, then two identities in Lemma hold without the assumption that T is strict. In this case,

$$Wf(x) = \mathsf{P}^x \int_{]0,T[} f(X_t)d(-M_t);$$
$$W'f(x) = \mathsf{P}^x \int_{]0,T[} f(X_t)d[M]_t.$$

2) From the proof we can see that the identity (2.3.4) holds with the weaker assumption that $M \in \mathrm{MF}_+$.

Theorem 2.3.1 Let $u \in \mathscr{S}$ and T be a strict terminal time. Then we have

$$L^M(m, P_T^M u) = L(m, P_T u) + \rho_M^{R_T m}(P_T^M u). \tag{2.3.8}$$

Proof. Now suppose firstly that $m \in$ Con. Then $L(m, P_T u) = 0$ and by (2.2) of [12], we have $L^M(m, P_T^M u) = \rho_{[M]}^m(P_T^M u)$. Let $\phi(x) = \mathsf{P}^x(T < \infty)$. Then $R_T m = \phi \cdot m$ ((4.17) of [16]) and

$$\begin{aligned}
\rho_{[M]}^{R_T m}(P_T^M u) &= \lim_{t\to 0} \frac{1}{t} \mathsf{P}^{\phi m}\{1_{\{T<\infty\}} \int_0^t P_T^M u(X_s)d[M]_s\} \\
&= \lim_{t\to 0} \frac{1}{t} \mathsf{P}^{\phi m} \int_0^t [u(X_T)M_T 1_{\{T<\infty\}}]\circ\theta_s \, d[M]_s \\
&= \lim_{t\to 0} \frac{1}{t} \mathsf{P}^{\phi m}\{1_{\{T<\infty\}} \int_0^t [u(X_T)M_T 1_{\{T<\infty\}}]\circ\theta_s \, d[M]_s\} \\
&= \lim_{t\to 0} \frac{1}{t} \mathsf{P}^{m} \int_0^t P_T^M u(X_s)d[M]_s \\
&= \rho_{[M]}^m(P_T^M u).
\end{aligned}$$

Hence (2.3.8) holds for $m \in$ Con. If $m \in$ Dis, then there exists $\{\mu_n\}$ such that $\mu_n U \uparrow m$. Define $\nu_n := \mu_n + \mu_n U_{[M]}$. Then $\nu_n V = \mu_n(V + U_{[M]}V) = \mu_n U \uparrow m$ and using the identity (2.3.5),

$$\begin{aligned}
L^M(m, P_T^M u) &=\uparrow \lim_{n\uparrow\infty} \nu_n P_T^M u \\
&=\uparrow \lim_{n\uparrow\infty} (\mu_n + \mu_n U_{[M]}) P_T^M u
\end{aligned}$$

$$
\begin{aligned}
&=\uparrow \lim_{n\uparrow\infty} \mu_n(P_T^M + U_{[M]}P_T^M)u \\
&=\uparrow \lim_{n\uparrow\infty} \mu_n(P_T + P_T U_{[M]}P_T^M)u \\
&= L(m, P_T u) + L(m, P_T U_{[M]}P_T^M u) \\
&= L(m, P_T u) + \rho_{[M]}^{R_T m}(P_T^M u).
\end{aligned}
$$

Thus it follows that

$$
\begin{aligned}
L^M(m, P_T^M u) &= L(m, P_T u) + \rho_{[M]}^{R_T m}(P_T^M u) \\
&= L(m, P_T u) + \rho_M^{R_T m}(P_T^M u).
\end{aligned}
$$

That completes the proof. □

Remark. It follows directly from (2.3.8) that if $B \in \mathscr{E}^e$ and is strict,

$$
\Gamma^M(B) = \Gamma(B) + \rho_M^{R_B m}(P_B^M 1) \tag{2.3.9}
$$

which generalizes the formula (2.3.2).

Theorem 2.3.2 Let $B \in \mathscr{E}^e$ and be strict and $(\hat{P}^x, H)$ the exit system for B. Then

$$
\Gamma^M(B) = \Gamma(B) + \rho_M^{R_B m}(B^r) + \hat{P}^\rho[\frac{M_{T_B}}{M_{T_B -}} - M_{T_B};\ T_B < \infty] \tag{2.3.10}
$$

where $\rho := \rho_H^m$ and B^r is the set of regular points of B.

Proof. Using the formula (2.3.9), we have

$$
\begin{aligned}
\Gamma^M(B) &= \Gamma(B) + \rho_M^{R_B m}(P_B^M 1) \\
&= \Gamma(B) + \rho_M^{R_B m}(\mathsf{P}^{\cdot}(T_B = 0)) + \rho_M^{R_B m}(\mathsf{P}^{\cdot}(M_{T_B};\ 0 < T_B < \infty)) \\
&= \Gamma(B) + \rho_M^{R_B m}(B^r) + \rho_{[M]}^{R_B m}(\mathsf{P}^{\cdot}(M_T;\ 0 < T < \infty)),
\end{aligned}
$$

where $T := T_B$. Now we use the Kuznetsov measure and exit system formula (11.6) of [16] to compute the third term. Let τ be the extension of T on W and $T_s = T \circ \theta_s + s$. Take ϕ positive and $\int \phi(t)dt = 1$. Then

$$
\rho_{[M]}^{R_T m}(\mathsf{P}^{\cdot}(M_T; 0 < T < \infty))
$$

$$
\begin{aligned}
&= Q_{R_T m} \int_{-\infty}^{\infty} \phi(t) P^{Y_t}(M_T; 0 < T < \infty) d[M]_t^* \\
&= Q_{R_T m} \int \phi(t)(M_T)\circ\theta_t 1_{\{0<T\circ\theta_t<\infty\}} d[M]_t^* \\
&= Q_m \int \phi(t)(M_T)\circ\theta_t 1_{\{0<T\circ\theta_t<\infty;\ \tau<t\}} d[M]_t^* \\
&= Q_m \int \phi(t)[(M_T 1_{\{0<T<\infty\}})\circ\theta_0 1_{\{\tau<0\}}]\circ\sigma_t d[M]_t^* \\
&= Q_m \int \phi(t)(\sum_{s\in G^*} M_{T_s} 1_{\{s\le 0<T_s<\infty\}})\circ\sigma_t d[M]_t^* \\
&= Q_m \int \phi(t) \sum_{s+t\in G^*} M_{T_{s+t}-t}\circ\theta_t 1_{\{s\le 0<T_{s+t}-t<\infty\}} d[M]_t^* \\
&= Q_m \int \phi(t) \sum_{s\in G^*} M_{T_s-t}\circ\theta_t 1_{\{s\le t<T_s<\infty\}} d[M]_t^* \\
&= Q_m \sum_{s\in G^*} \int_{[s,T_s[} \phi(t) M_{T_t-t}\circ\theta_t 1_{\{T_s<\infty\}} d[M]_t^*
\end{aligned}
$$

(since $T_s = T_t$ for $t \in [s, T_s[$ and $s \in G^*$)

$$
\begin{aligned}
&= Q_m \sum_{s\in G^*} \int_{[0,T\circ\theta_s[} \phi(t+s)(M_T)\circ\theta_{s+t} 1_{\{T\circ\theta_s<\infty\}} d[M]_t\circ\theta_s \\
&= Q_m \sum_{s\in G^*} (\int_{[0,T[} \phi(s+t)(M_T)\circ\theta_t 1_{\{T<\infty\}} d[M]_t)\circ\theta_s \\
&= \hat{P}^\rho(\int ds \int_{[0,T[} \phi(s+t) \frac{M_{T\circ\theta_t+t}}{M_t} d[M]_t;\ T<\infty) \\
&= \hat{P}^\rho[M_T(\frac{1}{M_{T-}} - 1);\ T<\infty].
\end{aligned}
$$

That completes the proof. □

The following is an interesting switching identity generalizing (4.30) of [16].

Proposition 2.3.3 Let T be a strict terminal time. Then the following switching identity holds

$$\rho_M^{R_T^M m} P_T = \rho_M^{R_T m} P_T^M. \tag{2.3.11}$$

Proof. Replacing A by $[M]$ in (2.1.4), we have

$$U_{[M]}U_M = U_{[M]} - U_M. \tag{2.3.12}$$

It suffices to show (2.3.11) for $m \in \mathrm{Dis}$. There exist $\{\mu_n\}$ such that $\mu_n m \uparrow m$. Let $\nu_n = \mu_n + \mu_n U_{[M]}$. Then $\nu_n V \uparrow m$. Thus for $f \in p\mathscr{E}$, using (2.3.5) and (2.3.12),

$$\begin{aligned}
\rho_M^{R_T^M m} P_T f &= L^M(R_T^M m, U_M P_T f) \\
&=\uparrow \lim_{n\uparrow\infty} \nu_n P_T^M U_M P_T f \\
&=\uparrow \lim_{n\uparrow\infty} \mu_n (I + U_{[M]}) P_T^M U_M P_T f \\
&=\uparrow \lim_{n\uparrow\infty} \mu_n P_T (U_M + U_{[M]} P_T^M U_M) P_T f \\
&=\uparrow \lim_{n\uparrow\infty} \mu_n P_T (U_M + U_{[M]} (U_M - W)) P_T f \\
&=\uparrow \lim_{n\uparrow\infty} \mu_n P_T (U_{[M]} - U_{[M]} W) P_T f \\
&=\uparrow \lim_{n\uparrow\infty} \mu_n P_T (U_{[M]} P_T - U_{[M]} (P_T - P_T^M)) f \\
&=\uparrow \lim_{n\uparrow\infty} \mu_n P_T U_{[M]} P_T^M f \\
&= \rho_{[M]}^{R_T m} P_T^M f = \rho_M^{R_T m} P_T^M f.
\end{aligned}$$

That completes the proof. □

Now in addition we assume that M is continuous. Then $M_t = e^{-[M]_t}$ and $[M]$ is continuous. By the remark below the Lemma 2.3.1, the theorems above are true even without the assumption that T is strict. Moreover we have the following generalized form of the Getoor and Steffens' formula (2.3.3).

Theorem 2.3.4 Let $B \in \mathscr{E}^e$ be finely closed and $(\hat{\mathsf{P}}^x, H)$ the exit system of B. Then

$$\Gamma^M(B) = \Gamma(B) + \rho_M^m(B) + \int_0^\infty (1 - e^{-t}) \nu_B^M(dt) \tag{2.3.13}$$

where $\nu_B^M(dt) := \hat{P}^\rho([M]_{T_B} \in dt; T_B < \infty)$, a measure on $\mathbf{R}^+$, and $\rho := \rho_H^m$.

Proof. Clearly by Theorem 2.3.2 it suffices to show that

$$\rho_{[M]}^{R_B m}(B^r) = \rho_M^m(B).$$

Since B is finely closed and $[M]$ is continuous, $B^r \subset B$ and $\rho_{[M]}^{R_B m}$ does not charge the semi-polar set $B - B^r$. Thus

$$\rho_{[M]}^{R_B m}(B^r) = \rho_{[M]}^{R_B m}(B).$$

Now by the formula (8.21) of [16],

$$\begin{aligned}\rho_{[M]}^{R_B m}(B) &= Q_{R_B m} \int \phi(t) 1_{\{Y_t \in B\}} d[M]_t^* \\ &= Q_m \int \phi(t) 1_{\{Y_t \in B\} \cap \{t > \tau_B\}} d[M]_t^*,\end{aligned}$$

where τ_B is the extension of T_B. But $\{Y_t \in B\} \subset \{\tau_B \leq t\}$ and $[M]^*$ is also continuous,

$$\begin{aligned}\rho_{[M]}^{R_B m}(B) &= Q_m \int \phi(t) 1_{\{Y_t \in B\}} d[M]_t^* \\ &= \rho_{[M]}^m(B) = \rho_M^m(B).\end{aligned}$$

That completes the proof. □

Chapter 3

Bivariate Revuz Measures

In this chapter we are mainly study potential theory of multiplicative functionals of Markov processes under duality by using their bivariate Revuz measures. In some sense, this chapter may be viewed as a continuation of the last chapter. Under the setting of weak duality and some additional conditions, two results of Sharpe (representation of terminal times and decomposition of MFs) are given in §3.1. They are critical throughout this article. Sharpe's canonical measure and Lévy system, which are used to describe discontinuities of the process, are discussed in §3.2. They are employed to give a representation for bivariate Revuz measures in §3.3. A generalized Revuz formula is given in §3.4 and it is especially useful and indispensable in dealing with multiplicative functionals. In §3.5 we use Kuznetsov measures to prove that a multiplicative functional is uniquely determined by its bivariate Revuz measure and that two multiplicative functionals are dual if and only if their respective bivariate Revuz measures are dual. Finally in §3.6 a few switching identities are given to generalize well-known Revuz formula.
Notations: For $M \in \mathrm{MF}$, let $\overline{M}_t := 1 - M_t$. Clearly $\overline{M} \in \mathrm{AF}(M)$. For any process (Z_t) and increasing process (A_t), we define $(Z_- * A)_t := \int_0^t Z_{s-} dA_s$ and $(Z * A)_t := \int_0^t Z_s dA_s$. For any $f \in p\mathscr{E}$ and $F \in p\mathscr{E} \times \mathscr{E}$, define $f * A := f(X) * A$, $f_- * A := f(X_-) * A$ and $F * A := F(X_-, X) * A$, where $f(X) := (f(X_t))$, $f(X_-) := (f(X_{-t}))$ and $F(X_-, X) := (F(X_{t-}, X_t))$.

3.1 Decomposition and Weak* Duality

Let

$$X = (\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, X_t, \mathsf{P}^x)$$

and

$$\hat{X} = (\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathscr{F}}_t, \hat{\theta}_t, \hat{X}_t, \hat{\mathsf{P}}^x),$$

be two right Markov processes on $(E, \mathscr{E})$ with Borel sub-Markovian semigroups (P_t) and $(\hat{P}_t)$, respectively, and be in weak duality relative to a fixed σ-finite measure m on E: for all $f, g \in p\mathscr{E}$,

$$(P_t f, g) = (f, \hat{P}_t g), \quad t > 0, \tag{3.1.1}$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(m)$. Walsh [48] has shown that under weak duality,

$$X_{t-} \text{ exists in } E \text{ for all } t \in]0, \zeta[\text{ a.s. } \mathsf{P}^m, \tag{3.1.2}$$

and of course the dual assertion holds for $\hat{X}$. Assuming (3.1.2) identically on Ω (resp. on $\hat{\Omega}$) relieves us from carrying one more exceptional set everywhere. As is customary, we use the prefix "co" or "^" to describe quantities relative to $\hat{X}$. Though all definitions and standard results are stated regarding to X, the dual statements also apply to $\hat{X}$.

We will focus on a special class of multiplicative functionals which have a nice decomposition. From now on we will write *a.s.* for P^m-*a.s.* and drop the m on Revuz measures as they are taken with respect to m if no confusion would be caused.

Definition 3.1.1 (i) M, $N \in \mathrm{MF}(X)$ are m-equivalent provided that, for each $t > 0$, $M_t = N_t$ a.s. on $\{\zeta > t\}$. (ii) Let $M \in \mathrm{MF}$ and $A, B \in \mathrm{RAF}(M)$. Then A and B are m-equivalent provided that, for each $t > 0$, $A_t = B_t$ a.s. on $\{\zeta > t\}$.

In view of right continuity of MFs, this is equivalent to the statement that $t \longmapsto M_t$ and $t \longmapsto N_t$ are identical functions a.s. on $[0, \zeta[$. Hence equality between MFs (resp. M-RAFs) will always be understood to mean m-equivalence.

Theorem 3.1.1 If the following condition holds

(3.1a) every m-semipolar set is m-polar,

then any $M \in \mathrm{MF}_+$ has a decomposition

$$M_t = \prod_{0<s\le t}(1-\Phi(X_{s-},X_s))\exp\{-\int_0^t a(X_s)dA_s\}1_{[0,J_B)}(t), \tag{3.1.3}$$

where $\Phi \in \mathscr{E}\times\mathscr{E}$, $0 \le \Phi < 1$, Φ vanishes on the diagonal D of $E\times E$, $a \in p\mathscr{E}$, A is a continuous additive functinal of X, B is a Borel subset of $E\times E$ which is disjoint from D and $S_M = J_B := \inf\{t > 0 : (X_{t-}, X_t) \in B\}$.

The proof mimicks that of Theorem 7.1 of [38]. We will just sketch it here. The Stieltjes logarithm of M

$$(\mathrm{slog}M)_t := \int_0^t 1_{\{s<S_M\}}\frac{d(-M_s)}{M_{s-}} \tag{3.1.4}$$

is a S_M-additive functional which can be decomposed into the sum of a natural part and a purely discontinuous quasi-left-continuous part. Note that from (16.21) of [23] two equivalent statements of (3.1a) are

(3.1b) every natural additive functional is a.s. continuous;

(3.1c) if T is a thin natural terminal time, $\mathsf{P}^m(T<\zeta)=0$.

Hence the natural part is continuous and it follows from (16.14) of [23] that $S_M = J_B$ with $B \subset E\times E$ disjoint from D. The purely discontinuous part has common discontinuities as X and must be of the form $\sum_{s\le t}\Phi(X_{s-},X_s)$ with $0\le\Phi<1$ since all jumps of slog$M$ are less than 1.

The weak duality together with (3.1a) is called weak*duality. An example of weak*duality is a pair of nearly symmetric Markov processes (i.e., they satisfy the sector condition) in weak duality. In fact, it is proved in [10] that in this case every semipolar set is m-polar and m-polar is the same as m-copolar. Hence these two processes are in weak*duality.

Now $M \in \mathrm{MF}_+$ is said to be simple if it has the decomposition (3.1.3), in which case we write $M \in \mathrm{SMF}_+$. A terminal time $T \in \mathrm{SMF}_+$ means that $T = J_B > 0$ a.s. with $B \in \mathscr{E}\times\mathscr{E}$ which is disjoint from D.

3.2 Lévy Systems and Canonical Measures

A Lévy system for X is a pair (N, H), where N is a kernel on $(E, \mathscr{E}^u)$ with $N(x, \{x\}) = 0$ for any $x \in E$ and H is a continuous AF of X having a bounded 1-potential, such that for any non-negative $F \in \mathscr{E} \times \mathscr{E}$ vanishing on the diagonal and any predictable process Y,

$$\mathsf{P}^x \sum_{0<s\leq t} Y_s F(X_{s-}, X_s) = \mathsf{P}^x \int_0^t Y_s NF(X_s) dH_s, \tag{3.2.1}$$

where $NF(x) := \int F(x, y)N(x, dy)$. Refer to §73[37] for the existence of Lévy systems.

Lemma 3.2.1 Let $B \in \mathscr{E} \times \mathscr{E}$ and $T := J_B > 0$ a.s. Then $(X_{T-}, X_T) \in B$ a.s. on $\{T < \infty\}$ and

$$1_{\{t<T\}} = \prod_{s\leq t} 1_{B^c}(X_{s-}, X_s) \tag{3.2.2}$$

a.s. on Ω.

Proof. Define the second hit $T^2 := T\circ\theta_T + T$. The statement (3.2.2) is obviously true on $\{T = \infty\}$. On $\{T < \infty\}$, since $T > 0$ a.s., $T^2 - T = T\circ\theta_T > 0$ a.s. Hence $(X_{T-}, X_T) \in B$ a.s. on $\{T < \infty\}$ by the definition of J_B. Then (3.2.2) follows directly. □

Theorem 3.2.1 Let (N, H) be a Lévy system for X and $M \in \mathrm{SMF}_+$ with the decomposition (3.1.3). Define $\Psi := 1_B + 1_{B^c} \cdot \Phi$ and $N_o := (1 - \Psi(x, y))N(x, dy)$. Then (N_o, H) is a Lévy system of the M-subprocess (X, M).

Proof. We have

$$\begin{aligned} Q^x \sum_{0<s\leq t} Y_s F(X_{s-}, X_s) &= \mathsf{P}^x \sum_{0<s\leq t} Y_s F(X_{s-}, X_s) M_s \\ &= \mathsf{P}^x \sum_{0<s\leq t} Y_s M_{s-} F(X_{s-}, X_s) \frac{M_s}{M_{s-}}. \end{aligned}$$

By Theorem 3.2.1 the discontinuous part M^d of M takes the following form

$$M_s^d = \prod_{0<r\leq s} (1-\Phi)\cdot 1_{B^c}(X_{r-}, X_r). \tag{3.2.3}$$

Hence

$$\frac{M_s}{M_{s-}} = \frac{M_s^d}{M_{s-}^d} = (1-\Phi)\cdot 1_{B^c}(X_{s-}, X_s) = (1-\Psi)(X_{s-}, X_s)$$

and then

$$\begin{aligned} Q^x \sum_{0<s\leq t} Y_s F(X_{s-}, X_s) &= \mathsf{P}^x \sum Y_s M_{s-}[F(1-\Psi)](X_{s-}, X_s) \\ &= \mathsf{P}^x \int_0^t Y_s M_{s-} NF(1-\Psi)(X_s) dH_s \\ &= \mathsf{P}^x \int_0^t Y_s M_s NF(1-\Psi)(X_s) dH_s \\ &= Q^x \int_0^t Y_s N_o F(X_s) dH_s. \end{aligned}$$

By definition, (N_o, H) is a Lévy system of (X, M). □

We will next introduce the canonical measure of X relative to m, which was first used in [39]. While the Lévy system describes the discontinuities of the process completely, the canonical measure does this relative to m.

The canonical measure ν is characterized as a σ-finite measure on $E\times E$ which is carried by $E\times E - D$ and such that for any $F \in p\mathscr{E}\times\mathscr{E}$ vanishing on D

$$\nu(F) = \lim_{t\to 0} \frac{1}{t} \mathsf{P}^m \sum_{s\leq t} F(X_{s-}, X_s). \tag{3.2.4}$$

It follows from (3.2.1) that

$$\nu(dx, dy) = N(x, dy)\rho_H(dx). \tag{3.2.5}$$

where (N, H) is a Lévy system of X. Thus Theorem 3.2.1 implies:

Corollary 3.2.1 Let $M \in \text{SMF}_+$ and ν^M denote the canonical measure of (X, M) (relative to m). Then $\nu^M = (1 - \Psi)\nu$.

The transform by a multiplicative functional $M \in \text{MF}$ consists of two steps of killing: a first entrance time $R' = D_{E_M^c}$ and a multiplicative functional in $\text{MF}_+(R')$. The following result gives us a Lévy system of the R'-subprocess of X.

Theorem 3.2.2 Let (N, H) be a Lévy system of X. Then (N', H) is a Lévy system of the R'-subprocess of X, where $N'(x, dy) = 1_{E_M \times E_M}(x, y) \cdot N(x, dy)$.

Proof. Let $(\mathsf{P}_r^x : x \in E)$ be the probability distribution of the R'-subprocess. For any $F \in p\mathscr{E} \times \mathscr{E}$ which is supported on $E_M \times E_M$, a predictable process (Z_t) and $x \in E_M$

$$
\begin{aligned}
&\mathsf{P}_r^x(\sum_{s \le t} F(X_{s-}, X_s) Z_s) \\
&= \mathsf{P}^x(\sum_{s \le t} F(X_{s-}, X_s) Z_s 1_{]0,R'[}(s)) \\
&= \mathsf{P}^x(\sum_{s \le t} F(X_{s-}, X_s) Z_s 1_{]0,R']}(s)) - \mathsf{P}^x(F(X_{R'-}, X_{R'}); R' \le t) \\
&= \mathsf{P}^x \int_0^{t \wedge R'} Z_s N F(X_s) dH_s \\
&= \mathsf{P}^x \int_0^t Z_s 1_{[0,R'[}(s) N F(X_s) dH_s \\
&= \mathsf{P}^x \int_0^t Z_s 1_{[0,R'[}(s) N' F(X_s) dH_s \\
&= \mathsf{P}_r^x \int_0^t Z_s N' F(X_s) dH_s.
\end{aligned}
$$

The third equality holds since $F(X_{R'-}, X_{R'}) = 0$ due to the fine closeness of E_M^c and the fact that $X_{R'} \in E_M^c$. □

3.3 Representation of Bivariate Revuz Measures

Bivariate Revuz measures were also introduced in [38] and will play a critical role in coming sections. We assume that (N, H) is a Lévy system of X and $\nu(dx, dy) = N(x, dy)\rho_H(dx)$ the canonical measure of X relative to m in the sequel.

Definition 3.3.1 Let $M \in \mathrm{MF}$ and $A \in \mathrm{RAF}(M)$. The bivariate Revuz measure of A relative to m is defined to be

$$\nu_A(F) :=\uparrow \lim_{t\downarrow 0} \frac{1}{t}\mathsf{P}^m \int_0^t F(X_{s-}, X_s)dA_s, \quad F \in p\mathscr{E} \times \mathscr{E}. \tag{3.3.1}$$

The Revuz measure and left Revuz measure of A relative to m are defined to be

$$\begin{aligned}\rho_A(f) &:=\uparrow \lim_{t\downarrow 0} \frac{1}{t}\mathsf{P}^m \int_0^t f(X_s)dA_s;\\ \lambda_A(f) &:=\uparrow \lim_{t\downarrow 0} \frac{1}{t}\mathsf{P}^m \int_0^t f(X_{s-})dA_s, \quad f \in p\mathscr{E}.\end{aligned} \tag{3.3.2}$$

As is customary, $F(x, y) = 0$ if either $X = \Delta$ or $y = \Delta$. The existence of the limits above can be shown similarly as in [FG2 (2.5)]. Note that we need to assume that $X_{\zeta-}$ exists in (4.3) if A does charge ζ. For any $f, g \in \mathscr{E}$, we write $f(x)g(y)$ as $f \otimes g$. Clearly $\nu_A(1 \otimes \cdot) = \rho_A$, but $\nu_A(\cdot \otimes 1) \neq \lambda_A$ in general. We denote the bivariate Revuz measure, Revuz measure and left Revuz measure of $\overline{M}$ by ν_M, ρ_M and λ_M, respectively. We also define the bivariate potential of $A \in \mathrm{RAF}(M)$ by

$$\mathscr{U}_A^q F(x) := \mathsf{P}^x \int_0^\infty e^{-qt} F(X_{t-}, X_t)dA_t \tag{4.4}$$

for any $F \in p\mathscr{E} \times \mathscr{E}$ and $\mathscr{U}_M^q := \mathscr{U}_{\overline{M}}^q$ for $M \in \mathrm{MF}$. Denote $U_A^q := \mathscr{U}_A^q(1 \otimes \cdot)$ for $A \in \mathrm{RAF}(M)$, $U_M^q := \mathscr{U}_M^q(1 \otimes \cdot)$, and $P_M^q f(x) := U_M^q f(x)$ for $x \in E_M$; $P_M^q f(x) := f(x)$ for $x \notin E_M$. Clearly $P_M^q = U_M^q$ if $M \in \mathrm{MF}_+$. Recall that $[M]$ denotes the Stieltjes logarithm of $M \in \mathrm{MF}_{++}$ and in this case $[M] \in \mathrm{AF}$. The following lemma is just a collection of facts about bivariate Revuz measures. They can be checked either easily or by mimicking the proofs of the respective results for Revuz measures in [49] and also in chapter 2.

Lemma 3.3.1 Let $M \in \mathrm{MF}$ and $A \in \mathrm{RAF}(M)$.

(3.3.1a) $\nu_A(F) = \uparrow \lim_q qm\mathscr{U}_A^q F$;

(3.3.1b) $\mathscr{U}_A^q F \in \mathscr{S}^q(M)$ and $\nu_A(F) = L^M(m, \mathscr{U}_A F)$ if $m \in \mathrm{Dis}(M)$;

(3.3.1c) ν_A does not charge any m-bipolar set, which is a set $B \subset E \times E$ such that either $B \subset E \times B_0$ for some m-polar set B_0 or $B \subset B_1 \times E$ for some m-leftpolar set B_1;

(3.3.1d) $\lambda_A = \nu_A(\cdot \otimes 1)$ if A does not charge ζ;

(3.3.1e) If A is natural, $\nu_A(F) = \rho_A(F_D) = \lambda_A(F_D)$ for any $F \in p\mathscr{E} \times \mathscr{E}$, where $F_D(x) := F(x,x)$.

(3.3.1f) If $M \in \mathrm{MF}_{++}$, then $\nu_M = \nu_{[M]}$.

Now we are going to prove an important representation theorem for bivariate Revuz measures.

Theorem 3.3.1 Let $M \in \mathrm{SMF}_+$ with the decomposition (3.1.3). Then the bivariate Revuz measure of M is given by

(3.3.3) $$\nu_M(dx, dy) = (1_B + 1_{B^c} \cdot \Phi)(x,y) \cdot \nu(dx, dy) + \delta_{\{y\}}(dx) a(y) \rho_A(dy),$$

where $\delta_{\{y\}}$ is the point mass at y and ν is the canonical measure of X.

Proof. By Theorem 3.2.1, if $\Psi := 1_B + 1_{B^c} \cdot \Phi$, we have

(3.3.4) $$M_t = \big[\prod_{s \le t} (1 - \Psi)(X_{s-}, X_s)\big] \exp\big\{ - \int_0^t a(X_s) dA_s \big\}.$$

Define

$$M_t^d := \prod_{s \le t} (1 - \Psi)(X_{s-}, X_s);$$
$$M_t^c := \exp\big\{ - \int_0^t a(X_s) dA_s \big\} 1_{[0, S_M[}(t).$$

Clearly $M_t = M_t^d \cdot M_t^c$, $M_{S_M}^d = 0$ and $d(-M_s) = M_s^d d(-M_s^c) + M_{s-}^c d(-M_s^d)$. Then for $F \in p\mathscr{E} \times \mathscr{E}$, since A is continuous,

$$\begin{aligned}\int_0^t F(X_{s-}, X_s) M_s^d d(-M_s^c) &= \int_0^t F(X_{s-}, X_s) M_s^d 1_{\{s<S_M\}} d(-M_s^c) \\ &= \int_0^t F(X_{s-}, X_s) M_s a(X_s) dA_s \\ &= \int_0^t F_D(X_s) M_s a(X_s) dA_s.\end{aligned}$$

Hence by (2.22) of cite[FG]

$$\lim_{t\to 0} \frac{1}{t} \mathsf{P}^m \int_0^t F(X_{s-}, X_s) M_s^d d(-M_s^c) = \rho_A(F_D \cdot a). \tag{3.3.5}$$

On the other hand, since $M_{S_M}^d = 0$ and $\Psi(X_{S_M -}, X_{S_M}) = 1$,

$$\begin{aligned}&\int_0^t F(X_{s-}, X_s) M_{s-}^c d(-M_s^d) \\ &= \sum_{s\le t} F(X_{s-}, X_s) M_{s-}^c (M_{s-}^d - M_s^d) \\ &= \sum_{s\le t} F(X_{s-}, X_s) M_{s-}^c M_{s-}^d \cdot \Psi(X_{s-}, X_s).\end{aligned}$$

Then we use the Lévy system formula and (2.22)of [13]

$$\begin{aligned}&\lim_{t\to 0} \frac{1}{t} \mathsf{P}^m \int_0^t F(X_{s-}, X_s) M_{s-}^c d(-M_s^d) \\ &\quad= \lim_{t\to 0} \frac{1}{t} \mathsf{P}^m \int_0^t M_s N(F\Psi)(X_s) dH_s \\ &\quad= \int \rho_H(dx) N(F\Psi)(x) \\ &\quad= \Psi \cdot \nu(F).\end{aligned}$$

Hence we have

$$\nu_M(F) = \Psi \cdot \nu(F) + a \cdot \rho_A(F_D). \tag{3.3.6}$$

That proves (3.3.3) □

The following corollary is easy to check.

Corollary 3.3.1 Let $M \in \mathrm{SMF}_+$. Then we have

(a) $\nu_{S_M} = 1_B \cdot \nu$;

(b)

$$\nu^M + 1_{D^c} \cdot \nu_M = \nu.$$

Recall the definition of Stieltjes logarithm of a multiplicative functional in (3.1.4). Now we are going to compute its bivariate Revuz measure.

Proposition 3.3.2 Let $M \in \mathrm{SMF}_+$ with the decomposition (3.1.3). Then the bivariate Revuz measure of its Stieltjes logarithm slog$M$ is given by

$$\nu_{\mathrm{slog}M}(dx, dy) = 1_{B^c}\Phi(x, y) \cdot \nu(dx, dy) + \delta_{\{y\}}(dx)a(y)\rho_A(dy). \tag{3.3.7}$$

Proof. Since M admits the decomposition (3.1.3), its Stieltjes logarithm slog$M$ admits the following decomposition

$$(\mathrm{slog}M)_t = \sum_{s \le t} \Phi(X_{s-}, X_s)1_{\{s<S_M\}} + \int_0^t 1_{\{s<S_M\}}a(X_s)dA_s. \tag{3.3.8}$$

Hence for any $F \in p\mathscr{E} \times \mathscr{E}$, using the Lévy system formula, Theorem 3.2.1 and (2.22) of [13] we find

$$\begin{aligned}
\nu_{\mathrm{slog}M}(F) &= \uparrow \lim_{t\downarrow 0} \frac{1}{t}\mathsf{P}^m \int_0^t F(X_{s-}, X_s)d(\mathrm{slog}M)_t \\
&= \uparrow \lim_{t\downarrow 0} \frac{1}{t}\mathsf{P}^m \sum_{s\le t} F(X_{s-}, X_s)\Phi(X_{s-}, X_s)1_{\{s<S_M\}} \\
&\quad + \uparrow \lim_{t\downarrow 0} \frac{1}{t}\mathsf{P}^m \int_0^t F(X_{s-}, X_s)1_{\{s<S_M\}}a(X_s)dA_s \\
&= \uparrow \lim_{t\downarrow 0} \frac{1}{t}\mathsf{P}^m \sum_{s\le t} F(X_{s-}, X_s)\Phi(X_{s-}, X_s)\prod_{r\le s} 1_{B^c}(X_{s-}, X_s) + \rho_A(a \cdot F_D) \\
&= \uparrow \lim_{t\downarrow 0} \frac{1}{t}\mathsf{P}^m \int_0^t N(1_{B^c}\Phi F)(X_s)1_{\{s<S_M\}}dH_s + a\rho_A(F_D) \\
&= 1_{B^c} \cdot \Phi \cdot \nu(F) + a \cdot \rho_A(F_D).
\end{aligned}$$

That completes the proof. □

Combining (3.3.3), (3.3.1a) and (3.3.7), we can see that if $M \in \mathrm{SMF}_+$, then

$$\nu_M = \nu_{\mathrm{slog}M} + \nu_{S_M}. \tag{3.3.9}$$

But (3.3.9) holds even without the assumption that M has a simple decomposition. Actually we have a more general formula. Recall a useful formula in [49] or (2.2.8). If $M \in \mathrm{MF}_+$, $m \in \mathrm{Dis}$ and $u \in \mathscr{S}$, then

$$L(m, u) = L^M(m, u - P_M u). \tag{3.3.10}$$

Theorem 3.3.3 (i) Let $M, N \in \mathrm{MF}$ with $E_N \subset E_M$. Then

$$\nu_{MN}^{m^*} = {}^N\nu_M^{m^*} + \nu_N^{m^*},$$

where ${}^N\nu_M^{m^*}$ is the bivariate Revuz measure of M relative to $m^* = m|_{E_N}$ under the N-subprocess of X. (ii) Let $M \in \mathrm{MF}$. Then

$$\nu_M^{m^*} = \nu_{\mathrm{slog}M}^{m^*} + \nu_{S_M}^{m^*}.$$

Proof. (i) Clearly $E_{MN} = E_N$. For any $F \in p\mathscr{E} \times \mathscr{E}$,

$$\begin{aligned}\nu_{MN}^{m^*}(F) &= \lim_t \frac{1}{t}\mathsf{P}^{m^*}\int_0^t F(X_{s-}, X_s)d(-M_sN_s)\\ &= \lim_t \frac{1}{t}\mathsf{P}^{m^*}\int_0^t F(X_{s-}, X_s)[N_sd(-M_s) + M_{s-}d(-N_s)].\end{aligned}$$

But

$$\mathsf{P}^{m^*}\int_0^t F(X_{s-}, X_s)N_sd(-M_s) = P_{(N)}^{m^*}\int_0^t F(X_{s-}, X_s)d(-M_s),$$

where $(P_{(N)}^x)$ denotes the probability measures corresponding to the N-subprocess of X. Thus if we write $(M_- * N)_t := M_- * \overline{N}$, then

$$\nu_{MN}^{m^*} = {}^N\nu_M^{m^*} + \nu_{M_-*N}^{m^*}.$$

We claim that the following identity holds

$$\mathscr{U}_N = P_M^{(N)}\mathscr{U}_N + \mathscr{U}_{M_-*N} \text{ on } E_N, \tag{3.3.11}$$

where $P_M^{(N)}f(x) := \mathsf{P}^x_{(N)}\int_0^\infty f(X_t)d(-M_t) = \mathsf{P}^x\int_0^\infty f(X_t)N_t d(-M_t)$. In fact for $F \in p\mathscr{E}\times\mathscr{E}$ and $x \in E_N$ with $\mathscr{U}_N F(x) < \infty$

$$\begin{aligned} P_M^{(N)}\mathscr{U}_F(x) &= \mathsf{P}^x_{(N)}\int_0^\infty \mathscr{U}_N F(X_t)d(-M_t) \\ &= \mathsf{P}^x\int_0^\infty \Big(\int_0^\infty F(X_{s-},X_s)d(-N_s)\Big)\circ\theta_t d(-M_t) \\ &= \mathsf{P}^x\int_0^\infty\int_t^\infty F(X_{s-},X_s)d(-N_s)d(-M_t) \\ &= \mathscr{U}_N F(x) - \mathscr{U}_{M-*N}F(x). \end{aligned}$$

Since $M \in \mathrm{MF}_=(X,N)$, we can use (3.3.10) and find

$$\nu_N^{m^*}(F) = L^N(m^*,\mathscr{U}_N F) = L^{MN}(m^*,\mathscr{U}_{M-*N}F) = \nu^{m^*}_{M-*N}(F).$$

(ii) Let $N = 1_{[0,S_M[}$. Then $M = MN$, $E_N = E_M$ and by (i) we have $\nu_M^{m^*} = {}^N\nu_M^{m^*} + \nu_{S_M}^{m^*}$. but

$${}^N\nu_M^{m^*}(F) = \lim_t \frac{1}{t}\mathsf{P}^{m^*}\int_0^t F(X_{s-},X_s)1_{\{s<S_M\}}d(-M_s) = \nu^{m^*}_{M-*\mathrm{slog}M}.$$

Following (3.3.11) and using the fact that $\mathrm{slog}M \in \mathrm{AF}(S_M)$, it is easy to check that

$$\mathscr{U}_{\mathrm{slog}M} = P_M^{(S_M)}\mathscr{U}_{\mathrm{slog}M} + \mathscr{U}_{M-*\mathrm{slog}M} \text{ on } E_M.$$

Then using (3.3.10) again, we have

$${}^N\nu_M^{m^*} = \nu^{m^*}_{M-*\mathrm{slog}M} = \nu^{m^*}_{\mathrm{slog}M}.$$

That completes the proof. Our last result of this section describes the relationship between the left Revuz measure λ_M and left marginal measure of ν_M.

Proposition 3.3.4 Assume that $X_{\zeta-}$ exists in E a.s. and $M_\zeta = 0$. Let $M \in \mathrm{MF}_+$ and κ be the killing measure of X relative to m (the left Revuz measure of ζ)

$$\kappa(f) :=\uparrow \lim_{t\to 0}\frac{1}{t}\mathsf{P}^m(f(X_{\zeta-});\zeta \le t).$$

Then $\lambda_M = \nu_M(\cdot\otimes 1) + \kappa$.

Proof. For any $f \in p\mathscr{E}$ we have

$$\begin{aligned}\lambda_M(f) &=\uparrow \lim_{t\to 0}\int_0^t f(X_{s-})d(-M_s)\\ &= \nu_M(f\otimes 1)+\uparrow \lim_{t\to 0}\frac{1}{t}\mathsf{P}^m(f(X_{\zeta-})M_{\zeta-};\zeta\le t).\end{aligned}$$

The second term is the left Revuz measure of $(\int_0^t M_{s-}d1_{[0,\zeta[}(s)) \in \mathrm{AF}(M)$, which is equal to κ by an argument similar to that used to prove (3.3.3). □

3.4 Generalized Revuz Formula

The Revuz formula was first given in [34] under strong duality and later proved by in [23] under weak duality. We quote the following form of Revuz formula from (9.9) of [23]. Let $A \in \mathrm{RAF}(X)$ which may charge the life time ζ. Assume that A has a σ-finite left Revuz measure, which is defined as

$$\lambda_A^m(f) := \lim_{t\to 0}\frac{1}{t}\mathsf{P}^m\int_0^t f(X_{s-})dA_s.$$

Hypothesis(H3.4). $X_{\zeta-}$ exists in E a.s. on $\{\Delta A_\zeta > 0\}$. Under (H3.4) it holds that

$$\lambda_A^m\hat{U}^q(dx) = (\mathsf{P}^x\int_0^\infty e^{-qt}dA_t)m(dx).$$

For any $f, g \in p\mathscr{E}$, and we have

$$(g, U_A^q f) = \lambda_{f*A}\hat{U}^q f = \nu_A(\hat{U}^q g\otimes f), \tag{3.4.1}$$

where $(f*A)_t := \int_0^t f(X_s)dA_s$. Note that $f*A$ never charges ζ and thus (3.4.1) holds without (H3.4).

In this section we will deduce a similar formula for an M-additive functional. Let $M \in \mathrm{MF}(X)$ and $A \in \mathrm{RAF}(M)$ having a σ-finite left Revuz measure relative to $1_{E_M}\cdot m \in \mathrm{Exc}(M)$, and $\hat{M}$ the dual multiplicative functional of M; namely, $\hat{M} \in \mathrm{MF}(\hat{X})$ such that (X, M) and $(\hat{X}, \hat{M})$ are in weak duality with respect to m. Write $m^* := m|_{E_M}$ and let (Q_t) and (V^q) (resp.,

$(\hat{Q}_t)$ and $(\hat{V}^q)$) be the transition semigroup and resolvent of (X, M) (resp., $(\hat{X}, \hat{M})$). Here A may charge ζ.

Now we will give the some preliminary lemmas and the generalized Revuz formula. The proofs we present here are analogous to those in [23].

Lemma 3.4.1 Let $\phi \in \mathscr{R}^+$ and let η be σ-finite measure on E. Then for each $t > 0$,

$$\int_0^\infty \phi(s)ds\, \mathsf{E}^{\eta Q_s}(f(X_0)A_t) = \mathsf{E}^\eta \int_0^\infty dA_r \int_0^\infty f(X_s)\phi(s)1_{[r-t,r)}(s)ds. \tag{3.4.2}$$

Proof. The direct computation gives

$$\begin{aligned}
\int_0^\infty \phi(s)ds\mathsf{E}^{\eta Q_s}(f(X_0)A_t) &= \int \phi(s)ds \int_E \int_E \eta(dx)Q_s(x,dy)\mathsf{E}^y(f(X_0)A_t) \\
&= \int \phi(s)ds \int \eta(dx)\mathsf{E}^x((\mathsf{E}^{X_s}f(X_0)A_t)M_s) \\
&= \int \phi(s)ds \int \eta(dx)\mathsf{E}^x[(f(X_0)A_t)\circ\theta_s M_s] \\
&= \int \phi(s)ds \int \eta(dx)\mathsf{E}^x f(X_s)(A_{t+s} - A_s) \\
&= \mathsf{E}^\eta \int_0^\infty dA_r \int_0^\infty f(X_s)\phi(s)1_{[r-t,r)}(s)ds.
\end{aligned}$$

That completes the proof. □

Theorem 3.4.1 Suppose that P^{m^*} a.s. $t \longmapsto f(X_{t-})$ is left continuous on $]0, \infty[$. Then under (H3.4)

$$\lim_{t\downarrow 0} \frac{1}{t}\mathsf{P}^{m^*} \int_0^t f(X_{s-})dA_s = \lim_{t\downarrow 0} \frac{1}{t}\mathsf{P}^{m^*}(f(X_0)A_t). \tag{3.4.3}$$

By the lemma above, the proof of this theorem is exactly the same as the proof of (8.7) of [23]. We won't repeat it here.

Theorem 3.4.2 Under (H3.4), the following Revuz formula holds,

$$\lambda_A^{m^*}\hat{V}^q(dx) = (\mathsf{E}^x\int_0^\infty e^{-qt}dA_t)m^*(dx). \tag{3.4.4}$$

Proof. Since $\lambda_A^{m^*}$ is σ-finite, it suffices to prove (3.4.4) when $\lambda_A^{m^*}(E) < \infty$. First notice that nothing will be changed if we replace m^* in the rightside of the above formula by m. Let $f \in \hat{S}^q(M)$ and be bounded. Then $t \longmapsto f(X_{t-})$ is left continuous on $]0, \infty[$ a.s. P^{m^*}. Let $C_t(x) = E^x(A_t)$. Thus $C_{t+s} = C_t + Q_tC_s$ and

$$\frac{1}{s}(f, C_{t+s} - C_s)_{m^*} = \frac{1}{s}(f, Q_tC_s)_{m^*} = \frac{1}{s}(\hat{Q}_tf, C_s)_{m^*} = \frac{1}{s}\mathsf{P}^{m^*}(\hat{Q}_tf(X_0)A_s)$$

By the previous theorem and the fact $e^{-qt}\hat{Q}_tf \in \hat{\mathscr{S}}^q(M)$, we have

$$\lim_{s\to 0}\frac{1}{s}(f, C_{t+s} - C_s) = \lim_{s\to 0}\frac{1}{s}\mathsf{P}^{m^*}\int_0^s \hat{Q}_tf(X_{r-})dA_r = \lambda_A^{m^*}(\hat{Q}_tf).$$

Therefore $\lambda_A^{m^*}(\hat{Q}_tf)$ is the right derivative of (f, C_t) and

$$(f, C_{t+s} - C_t) = (\hat{Q}_tf, C_s) \leq N \cdot (1, C_s) \leq N \cdot s\lambda_A^{m^*}(E) < \infty,$$

where $N := ||f||_\infty$. Thus $t \longmapsto (f, C_t)$ is absolutely continuous and it follows from $C_0 = 0$ that

$$(f, C_t) = \int_0^t \lambda_A^{m^*}(\hat{Q}_sf)ds. \tag{5.8}$$

For any bounded continuous $h \geq 0$ which supported by E_M, $qV^qh \longrightarrow h$. Hence (3.4.4) holds for such h. □

Finally we will present two useful consequences. Let $L(\hat{u}, v) := L(\hat{u}m, v)$ for $\hat{u} \in \hat{\mathscr{S}}$ and $v \in \mathscr{S}$. Since L is just the Dirichlet form of X in some sense, we thereby obtain some idea on transformations of Dirichlet forms.

Corollary 3.4.1 (1) The following generalized Revuz formula holds without (H3.4)

$$(3.4.5) \qquad (g, U_A^q f) = \nu_A^{m^*}(\hat{V}^q g \otimes f) \quad f, g \in p\mathscr{E}.$$

(2) If $M \in$ MF and $S_M > 0$ a.s., then

$$(3.4.6) \qquad L^M(\hat{u}, v) = L(\hat{u}, v) + \nu_M^m(\hat{u} \otimes v) \ \ for\ \hat{u} \in \hat{\mathscr{S}},\ v \in \mathscr{S}.$$

Proof. Part (1) is immediate from Theorem 3.4.2 as we indicated in (3.4.1). (2) Since $\hat{u}m \in$Exc(X) for $\hat{u} \in \hat{\mathscr{S}}$, using (2.5) of [49] or (2.1.3) and Theorem 3.4.1, we have

$$\begin{aligned} L^M(\hat{u}, v) &= L(\hat{u}, v) + \rho_M^{\hat{u}m}(v) \\ &= L(\hat{u}, v) + \lim_{t \to 0} \frac{1}{t} \mathsf{P}^m(\hat{u}(X_0) \int_0^t v(X_s) d(-M_s)) \\ &= L(\hat{u}, v) + \lim_{t \to 0} \frac{1}{t} \mathsf{P}^m \int_0^t \hat{u}(X_{s-}) v(X_s) d(-M_s) \\ &= L(\hat{u}, v) + \nu_M^m(\hat{u} \otimes v). \end{aligned}$$

The reason that we can apply (3.4.3) here is that $(\int_0^t v(X_s)d(-M_s))_{t \geq 0}$ does not charge ζ. □

3.5 Uniqueness and dual multiplicative functionals

The following lemma is an m-a.e. version of (IV.2.12) of [3].

Lemma 3.5.1 Let $A^1, A^2 \in$ AF. Suppose that, for some fixed $q > 0$, $U_{A^1}^q 1 < \infty$ a.e. m, and that for any bouned and continuous $f \in p\mathscr{E}$, $U_{A^1}^q f = U_{A^2}^q f$ a.e. m. Then A^1 and A^2 are m-equivalent.

Proof. It is clear that a.e. m above can be replaced by q.e., namely, except a set of m-capacity zero. It follows from the proof of IV(2.12) of [3] that $U_{A^1}^p f = U_{A^2}^p f$ a.e. m for $p \geq q$. Since $q \longrightarrow U_{A^i}^q f(x)$ is decreasing and continuous for any $x \in E$, there exists $N \in \mathscr{E}$ with $m(N) = 0$ such that

$U^p_{A^i}1(x) < \infty$ and $U^p_{A^1}f(x) = U^p_{A^2}f(x)$ hold for any $p \geq q$ and $x \notin N$. Thus it follows that A^1 and A^2 are m-equivalent. □

First we have the following uniqueness result.

Proposition 3.5.1 Suppose that $A^1, A^2 \in$ AF which are σ-integrable. A^1 and A^2 are m-equivalent if and only if their bivariate Revuz measures are the same

$$\nu_{A^1} = \nu_{A^2}.$$

Proof. The **only if** part is trivial. Now assume that $\nu_{A^1} = \nu_{A^2}$. Since they are σ-finite, there exist a sequence of Borel functions $b_n \in \mathscr{E} \times \mathscr{E}$ such that

(1) $0 < b_n < 1$ for any n;

(2) $b_n \uparrow 1$;

(3) Both $b_n \cdot \nu_{A^1}$ and $b_n \cdot \nu_{A^2}$ are finite.

Let

$$A^{1,n}_t = \int_0^t b_n(X_{s-}, X_s)dA^1_s$$
$$A^{2,n}_t = \int_0^t b_n(X_{s-}, X_s)dA^2_s.$$

Thus

$$\nu_{A^{1,n}} = b_n \cdot \nu_{A^1} = b_n \cdot \nu_{A^2} = \nu_{A^{2,n}}.$$

From Lemma 3.5.1, it follows that $U^q_{A^{1,n}}f = U^q_{A^{2,n}}f$ a.e. m for any $f \in p\mathscr{E}$. On the other hands, by the Revuz formula,

$$(1, U^1_{A^{1,n}}1) = \nu_{A^{1,n}}(\hat{U}^1 1 \otimes 1) \leq \nu_{A^{1,n}}(1) < \infty.$$

Thus $U^1_{A^{1,n}}1 < \infty$ a.e. m. Therefore $A^{1,n}$ and $A^{2,n}$ are m-equivalent and by passing to limit, A^1 and A^2 are m-equivalent. □

Let $M, N \in$ MF and S_M, S_N the lifetimes of M, N, respectively. If M and N are m-equivalent, then P^m a.s. $S_M = S_N$, $E_M \triangle E_N$ is m-polar and $\nu^{m^*}_M = \nu^{m^*}_N$ where $m^* := m|_{E_M} = m|_{E_N}$. The following result is a partial answer to its inverse problem.

Theorem 3.5.2 Assume that $M, N \in \mathrm{SMF}_+$. Then M and N are m-equivalent if and only if $\nu_M = \nu_N$.

Proof. It suffices to check the sufficiency. Clearly M and N have the decompositions

$$M_t = [\prod_{s\le t}(1-\Phi)(X_{s-}, X_s)] \exp\big\{-\int_0^t a(X_s)dA_s\big\};$$
$$N_t = [\prod_{s\le t}(1-\Psi)(X_{s-}, X_s)] \exp\big\{-\int_0^t b(X_s)dB_s\big\},$$

where

1) $S_M = J_K > 0$, $S_N = J_L > 0$, and both K and L are disjoint from D;

2) $\Phi, \Psi \in p\mathscr{E} \times E$, $\Phi < 1$ and $\Psi < 1$ everywhere;

3) $a, b \in p\mathscr{E}$ and $A, B \in \mathrm{AF}$ being continuous.

Then we know that

$$\nu_M|_{D^c} = (1_{K^c}\Phi + 1_K)\cdot\nu, \quad \nu_N|_{D^c} = (1_{L^c}\Psi + 1_L)\cdot\nu.$$

Hence $\nu_M = \nu_N$ implies that

$$(1_{K^c}\Phi + 1_K)\cdot\nu = (1_{L^c}\Psi + 1_L)\cdot\nu.$$

Thus we have

$$1_{K^c}(1-\Phi)\cdot\nu = 1_{L^c}(1-\Psi)\cdot\nu.$$

Since $1-\Phi > 0$ and $1-\Psi > 0$ everywhere, $\nu(K^c \triangle L^c) = 0$, or equivalently $\nu(K \triangle L) = 0$.

Choose a sequence of Borel subsets (E_n) of $E \times E$ satisfying:

1) $E_n \uparrow E \times E - D$;

2) $\nu(E_n) < \infty$ for any n;

3) $E_n \subset \{(x, y): d(x, y) > \frac{1}{n}\}$, where d is a metric on E compatible with the given topology.

The existence of such sequence $\{E_n\}$ is easy to check. Let

$$S_{M,n} := J_{K\cap E_n} \quad \text{and} \quad S_{N,n} := J_{L\cap E_n}.$$

Clearly $\nu\{(K \cap E_n) \triangle (L \cap E_n)\} = 0$. Define

$$K_t^n := \sum_{s \le t} 1_{K \cap E_n}(X_{s-}, X_s) \ (= \sum_k 1_{\{S_{M,n}^k \le t\}})$$

and

$$L_t^n := \sum_{s \le t} 1_{L \cap E_n}(X_{s-}, X_s) \ (= \sum_k 1_{\{S_{N,n}^k \le t\}}),$$

where $S_{M,n}^k$ and $S_{N,n}^k$ are the k^{th}-iterates of $S_{M,n}$ and $S_{N,n}$.

Then $K^n, L^n \in \text{AF}$ and

$$\nu_{K^n} = 1_{K \cap E_n} \cdot \nu = 1_{L \cap E_n} \cdot \nu = \nu_{L^n}.$$

From the Lemma (6.2), we know that for any $f, g \in p\mathscr{E}$,

$$(f, U_{K^n}^1 g) = \nu_{K^n}(\hat{U}^1 f \otimes g) = \nu_{L^n}(\hat{U}^1 f \otimes g) = (f, U_{L^n}^1 g),$$

i.e., $U_{K^n}^1 g = U_{L^n}^1 g$ a.e. m.

On the other hands, by the Revuz formula,

$$(1, U_{K^n}^1 1) = \nu_{K^n}(\hat{U}^1 1 \otimes 1) \le \nu_{K^n}(1) = \nu(K \cap E_n) < \infty.$$

Hence $U_{K^n}^1 1 < \infty$ a.e. m. Therefore K^n and L^n are m-equivalent. But

$$S_{M,n} = \inf\{t > 0 : K_t^n > 0\} \ \text{ and } S_{N,n} = \inf\{t > 0 : L_t^n > 0\}.$$

It follows that P^m a.s. $S_{M,n} = S_{N,n}$ and thus P^m a.s. $S_M = S_N$.

Now we have $\text{slog}M$, $\text{slog}N \in \text{AF}(S)$, where $S := S_M = S_N$. But by (3.3.9)

$$\nu_{\text{slog}M} = \nu_{\text{slog}N}.$$

It follows form Proposition 3.5.1 that $\text{slog}M$ and $\text{slog}N$ are m-equivalent. Hence M and N are m-equivalent. □

Before our next result, we would like to recall some background. Two multiplicative functionals $M \in \text{MF}(X)$ and $\hat{M} \in \text{MF}(\hat{X})$ are said to be dual if their corresponding subprocesses are dual with respect to m. In [38], Sharpe proved that if X and $\hat{X}$ are in strong duality relative to m and

$M \in \mathrm{MF}(X)$ and $\hat{N} \in \mathrm{MF}(\hat{X})$, then M and $\hat{N}$ are dual if and only if, 1) the corresponding exact regularizations of S_M and $\hat{S}_{\hat{N}}$ are dual terminal times; 2) the bivariate Revuz measures of the Stieltjes logarithms of M and $\hat{N}$ are dual; that is

$$\nu_{\mathrm{slog} M}(dx, dy) = \hat{\nu}_{\mathrm{slog} \hat{N}}(dy, dx).$$

Here we are going to prove some similar results in weak duality by means of Kuznetsov measures.

Let Q be the Kuznetsov measure of X and m. Ω is identified as a subspace of W, i.e.,

$$\Omega = \{w \in W : \ \alpha(w) = 0, \ Y_{\alpha+}(w) \text{ exists in } E\}. \tag{3.5.1}$$

Shift operators $(\sigma_t)_{t \in \mathbf{R}}$ and truncated shift operators $(\theta_t)_{t \in \mathbf{R}}$ are defined as

$$\sigma_t w(s) := w(t+s) \ \text{ for any } t, s \in \mathbf{R},$$

$$\theta_t w(s) := w(t+s) \text{ if } s > 0, \ \theta_t w(s) := \Delta \text{ if } s \leq 0.$$

Clearly $\theta_t|_\Omega$ is the shift operator of X and $\theta(\{\alpha < t\}) \subset \Omega$. We also define a reversal operator $\lambda : \ W \to W$ as

$$\lambda w(s) := w((-s)-)$$

for any $s \in R$ (write $\hat{w} := \lambda w$ and $\hat{Y}_t(w) = Y_t(\hat{w})$) and the backward shift operator $\hat{\sigma}_t$ on W naturally as

$$\hat{\sigma}_t \hat{w}(s) := \hat{w}(t+s).$$

We have

$$\begin{aligned} \hat{\sigma}_t \circ \lambda w(s) &= \hat{\sigma}_t \hat{w}(s) = \hat{w}(t+s) \\ &= w((-t-s)-) = \sigma_{-t} w((-s)-) \\ &= \lambda \circ \sigma_{-t} w(s). \end{aligned}$$

Hence $\hat{\sigma}_t \circ \lambda = \lambda \circ \sigma_{-t}$. We also have $\hat{\alpha} := \alpha \circ \lambda = -\beta$ and $\hat{\beta} := \beta \circ \lambda = -\alpha$.

Similarly, $\hat{\Omega}$ is identified as

$$\hat{\Omega} := \{\hat{w} \in W : \ \alpha(\hat{w}) = 0, \ Y_{\alpha+}(\hat{w}) \text{ exists in } E\}$$

or equivalently,

$$w \in \Omega \Longleftrightarrow \lambda w \in \hat{\Omega}.$$

Let $\hat{Q}$ be the Kuznetsov measure of $\hat{X}$ and m on W, i.e., for $t_1 < t_2 < \cdots < t_n$,

$$\begin{aligned}&\hat{Q}(\alpha < t_1, Y_{t_1} \in dx_1, \ldots, Y_{t_n} \in dx_n, t_n < \beta)\\&= m(dx_1)\hat{P}_{t_2-t_1}(x_1, dx_2) \cdots \hat{P}_{t_n-t_{n-1}}(x_{n-1}, dx_n).\end{aligned}$$

It is known that $\hat{Q}$ is the reversal of Q, i.e.,

$$\hat{Q} = \lambda(Q) = Q\circ\lambda^{-1}. \tag{3.5.2}$$

In fact for any $t_1 < t_2 < \cdots < t_n$,

$$\begin{aligned}&\lambda(Q)(\alpha < t_1, Y_{t_1} \in dx_1, \ldots, Y_{t_n} \in dx_n, t_n < \beta)\\&= Q(\alpha < -t_n, Y_{-t_n} \in dx_n, \ldots, Y_{-t_1} \in dx_1, -t_1 < \beta)\\&= m(dx_n)P_{t_n-t_{n-1}}(x_n, dx_{n-1}) \cdots P_{t_2-t_1}(x_2, dx_1)\\&= \hat{P}_{t_n-t_{n-1}}(x_{n-1}, dx_n) \cdots \hat{P}_{t_2-t_1}(x_1, dx_2)m(dx_1)\\&= \hat{Q}(\alpha < t_1, Y_{t_1} \in dx_1, \ldots, Y_{t_n} \in dx_n, t_n < \beta).\end{aligned}$$

Let $A \in \mathrm{RAF}(X)$ which induces a HRM, denoted by A^*, of Y by the formula

$$A^*(w, B) :=\uparrow \lim_{t\downarrow\alpha(w)} A(\theta_t w, B - t). \tag{3.5.3}$$

Precisely, A^* is carried by $]\alpha, \beta]$ and

$$A^*(ds)\circ\sigma_t = A^*(ds + t).$$

Now define an RM on W

$$\hat{A}^*(\hat{w},]s, t[) := A^*(w,] - t, -s[). \tag{3.5.4}$$

Lemma 3.5.2 $\hat{A}^*$ is a backward HRM of Y in the sense

$$\hat{A}^*(\hat{\sigma}_t(\hat{w}), ds) = \hat{A}^*(\hat{w}, ds + t).$$

Proof. Check directly by definition,

$$\begin{aligned}\hat{A}^*(\hat{\sigma}_t(\hat{w}),]r, s[) &= \hat{A}^*(\hat{\sigma}_t \circ \lambda(w),]r, s[) \\ &= \hat{A}^*(\lambda \circ \sigma_{-t}(w),]r, s[) \\ &= A^*(\sigma_{-t}(w),] - s, -r[) \\ &= A^*(w,] - s - t, -r - t[) \\ &= \hat{A}^*(\hat{w},]s, r[+t).\end{aligned}$$

That completes the proof. □

Definition 3.5.1 1) Let $A \in \mathrm{RAF}(X)$ and $\hat{A} \in \mathrm{RAF}(\hat{X})$. They are said to be dual if $\hat{A}$ is m-equivalent to $\hat{A}^*|_{\hat{\Omega}}$ where $\hat{A}^*$ is defined in (3.5.4). 2) Two measures μ_1 and μ_2 on $E \times E$ are said to be dual if $\mu_1(dx, dy) = \mu_2(dy, dx)$.

Remark. The construction of the dual multiplicative functionals is similar. For example, a terminal time T of X induces a stationary terminal time T^* of Y by

$$T^*(w) :=\downarrow \lim_{t \downarrow \alpha(w)} (T(\theta_t w) + t).$$

Define the dual of T^* by

$$\hat{T}^*(\hat{w}) := -T^*(w) \text{ and set } \hat{T} := \hat{T}^*|_{\hat{\Omega}}.$$

Then $\hat{T}$ is the dual terminal time of T (See [Mi2]). Clearly the birth time α and the death time β are dual.

Theorem 3.5.3 1) Let $A \in \mathrm{AF}(X)$ and $\hat{A} \in \mathrm{AF}(\hat{X})$ which are σ-integrable. Then A and $\hat{A}$ are dual if and only if ν_A and $\hat{\nu}_{\hat{A}}$ are dual. 2) Furthermore if A and $\hat{A}$ are natural, then they are dual if and only if $\rho_A = \hat{\rho}_{\hat{A}}$.

Proof. 1) First assume A and $\hat{A}$ are dual. We can identify $\hat{A}$ here with the dual constructed above. Now by (8.10) of [16], for any $F \in p\mathscr{E} \times \mathscr{E}$,

$$\hat{\nu}_{\hat{A}}(F) = \hat{Q} \int \phi(t) F(Y_{t-}, Y_t) \hat{A}^*(dt)$$

$$= Q\int \phi(t)F(Y_{t-}(\hat{w}), Y_t(\hat{w}))\hat{A}^*(\hat{w}, dt)$$
$$= Q\int \phi(t)F(Y_{-t}(w), Y_{(-t)-}(w))A^*(w, -dt)$$
$$= Q\int \phi(-t)F(Y_t, Y_{t-})A^*(dt) = \nu_A(\hat{F})$$

where $\phi \in p\mathscr{R}$ and $\int \phi(t)dt = 1$ and $\hat{F}(x,y) := F(y,x)$.

Conversely, assume that ν_A and $\hat{\nu}_{\hat{A}}$ are dual. Let $\hat{A}' := \hat{A}^*|_{\hat{\Omega}}$ Then we have

$$\hat{\nu}_{\hat{A}'}(dx, dy) = \nu_A(dy, dx) = \hat{\nu}_{\hat{A}}(dx, dy).$$

From Proposition 3.5.1 it follows that $\hat{A}$ and $\hat{A}'$ are m-equivalent. Therefore A and $\hat{A}$ are dual. 2) follows from 1) directly, due to the fact that bivariate Revuz measures of natural additive functionals are carried by the diagonal D. □

Theorem 3.5.4 Let $M \in \mathrm{MF}_{++}(X)$ and $\hat{M} \in \mathrm{MF}_{++}(\hat{X})$. Then the following statements are equivalent.

1. M and $\hat{M}$ are dual;
2. $[M]$ and $[\hat{M}]$ are dual;
3. ν_M and $\hat{\nu}_{\hat{M}}$ are dual.

Proof. Since $\nu_M = \nu_{[M]}$ and $\hat{\nu}_{\hat{M}} = \hat{\nu}_{[\hat{M}]}$, the eqivalence of (2) and (3) follows from Theorem 3.5.3. Now we are going to prove the equivalence of (1) and (3). First we assume that M and $\hat{M}$ are dual. Using the Revuz formulas (3.4.1) and (3.4.5), we find for any bounded $f, g \in p\mathscr{E}$,

$$\nu_M(\hat{U}^1 g \otimes V^1 f) = \nu_{[M]}(\hat{U}^1 g \otimes V^1 f) = (g, U^1_{[M]} V^1 f)$$
$$= (\hat{U}^1_{[\hat{M}]} \hat{V}^1 g, f) = (f, \hat{U}^1_{\hat{M}} \hat{U}^1 g)$$
$$= \hat{\nu}_{\hat{M}}(V^1 f \otimes \hat{U}^1 g).$$

Let $\mu_1(dx) := \nu_M(dx \otimes V^1 f)$ and $\mu_2(dx) := \hat{\nu}_{\hat{M}}(V^1 f \otimes dx)$ for any fixed $f \in bp\mathscr{E}$. Then $\mu_1 \hat{U}^1 = \mu_2 \hat{U}^1$. Pick $h \in \mathscr{E}$ strictly positive such that

$m(h) < \infty$. Then $\mu_1 \hat{U}^1(h) = \mu_2 \hat{U}^1(h) = \nu_M(\hat{U}^1 h \otimes V^1 f) = (h, U^1_{[M]} V^1 f) \leq (h, U^1 f) < \infty$; that is, $\mu_1 \hat{U}^1$ and $\mu_2 \hat{U}^1$ are σ-finite. This implies that $\mu_1 = \mu_2$ by (2.12) of [16] or $\nu_M(dx \otimes V^1 f) = \hat{\nu}_{\hat{M}}(V^1 f \otimes dx)$. On the other hand, let $\nu_1(dy) := \nu_M(g \otimes dy)$ and $\nu_2(dy) := \hat{\nu}_{\hat{M}}(dy \otimes g)$ for any fixed $g \in bp\mathscr{E}$. Clearly $\nu_1 V^1 = \nu_2 V^1$ and, using the Revuz formula (3.4.5) again, $\nu_1 V^1(h) = \nu_2 V^1(h) = \hat{\nu}_{\hat{M}}(V^1 h \otimes g) = (\hat{U}^1_{\hat{M}} g, h) < \infty$; that is, $\nu_1 V^1$ and $\nu_2 V^1$ are σ-finite. Hence $\nu_1 = \nu_2$; i.e., ν_M and $\hat{\nu}_{\hat{M}}$ are dual.

Conversely, if ν_M and $\hat{\nu}_{\hat{M}}$ are dual, let $N \in \mathrm{MF}(X)$ denote the dual of $\hat{M}$. Then we have

$$\nu_M(dx, dy) = \hat{\nu}_{\hat{N}}(dy, dx) = \nu_N(dx, dy).$$

Hence $\nu_M = \nu_N$. By Proposition 3.5.1 M and N are m-equivalent. Therefore M and $\hat{M}$ are dual. □

Proposition 3.5.5 (1) Let $S \in \mathrm{SMF}_+(X)$ and $\hat{S} \in \mathrm{SMF}_+(\hat{X})$ be terminal times. Then S and $\hat{S}$ are dual if and only if ν_S and $\hat{\nu}_{\hat{S}}$ are dual. (2) The canonical measures ν and $\hat{\nu}$ of X and $\hat{X}$ are dual.

Proof. (1) First let S and $\hat{S}$ are dual. There exist Borel subsets $K, L \subset E \times E - D$ such that

$$\mathsf{P}^m \text{ a.s. } S = J_K \quad \text{and } \hat{\mathsf{P}}^m \text{ a.s. } \hat{S} = \hat{J}_L.$$

Let

$$G_n := \{(x, y) \in E \times E : \ d(x, y) > \frac{1}{n}\}.$$

Since $G_n = \hat{G}_n$, J_{G_n} and $\hat{J}_{G_n}$ are dual (see [19]). Set

$$S_n := J_{K \cap G_n} = S \wedge J_{G_n}$$

and

$$\hat{S}_n := \hat{J}_{L \cap \hat{G}_n} = \hat{S} \wedge \hat{J}_{G_n}.$$

Obviously, S_n and $\hat{S}_n$ are also dual terminal times. Let

$$A^n_t := \sum_{s \leq t} 1_{K \cap G_n}(X_{s-}, X_s) = \sum_k 1_{\{S^k_n \leq t\}}$$

$$\hat{A}^n_t := \sum_{s\le t} 1_{L\cap G_n}(\hat{X}_{s-}, \hat{X}_s) = \sum_k 1_{\{\hat{S}^k_n \le t\}},$$

where S^k_n, $\hat{S}^k_n$ are the corresponding k^{th}-iterate. Clearly A^n and $\hat{A}^n$ are dual. By Theorem 3.5.3

$$\begin{aligned}\nu_S|_{G_n}(dx, dy) &= \nu_{S_n}(dx, dy)\\ &= \nu_{A^n}(dx, dy) = \hat{\nu}_{\hat{A}^n}(dy, dx)\\ &= \hat{\nu}_{\hat{S}_n}(dy, dx) = \hat{\nu}_{\hat{S}}|_{G_n}(dy, dx).\end{aligned}$$

Since n is arbitrary and ν_S and $\hat{\nu}_{\hat{S}}$ do not charge D, it follows that ν_S and $\hat{\nu}_{\hat{S}}$ are dual.

Conversely if ν_S and $\hat{\nu}_{\hat{S}}$ are dual, let $\hat{K} := \{(x, y) : (y, x) \in K\}$ and $\hat{S}' := \hat{J}_{\hat{K}}$. Then S and $\hat{S}'$ are dual and we have

$$\hat{\nu}_{\hat{S}}(dx, dy) = \nu_S(dy, dx) = \hat{\nu}_{\hat{S}'}(dx, dy).$$

Hence $\hat{\nu}_{\hat{S}} = \hat{\nu}_{\hat{S}'}$. By Theorem 3.5.2 $\hat{S} = \hat{S}'$ and. S and $\hat{S}$ are dual.

(2) Using the notation above, since J_{G_n} and $\hat{J}_{G_n}$ are dual, their bivariate Revuz measures are dual; that is, $1_{G_n} \cdot \nu$ and $1_{G_n} \cdot \hat{\nu}$ are dual. Thus ν and $\hat{\nu}$ are dual. □

Corollary 3.5.1 Let $M \in \text{SMF}_+(X)$ and $\hat{M} \in \text{SMF}_+(\hat{X})$. Then M and $\hat{M}$ are dual if and only if ν_M and $\hat{\nu}_{\hat{M}}$ are dual.

Proof. Since duality of M and $\hat{M}$ implies duality of S_M and $\hat{S}_{\hat{M}}$ (see [4]), the necessity is a direct consequence of Theorem 3.5.4 and Proposition 3.5.5. Assume conversely that ν_M and $\hat{\nu}_{\hat{M}}$ are dual; that is,

$$(1_B + 1_{B^c} \cdot \Phi) \cdot \nu(dx, dy) = (1_{\hat{B}} + 1_{\hat{B}^c} \cdot \hat{\Phi}) \cdot \hat{\nu}(dy, dx),$$

where B and Φ (resp. $\hat{B}$ and $\hat{\Phi}$) are the parameters associated with the decomposition of M (resp. $\hat{M}$) as in (3.1.3). Since ν and $\hat{\nu}$ are dual, we find that ν_{S_M} and $\hat{\nu}_{\hat{S}_{\hat{M}}}$ are dual and that $\nu_{\text{slog}M}$ and $\hat{\nu}_{\text{slog}\hat{M}}$ are dual. Thus M and $\hat{M}$ are dual by heorem 3.5.4 and Proposition 3.5.5. □

3.6 Switching identities

In this section we shall present an identity which is a generalized form of what is usually called the Revuz formula. The results in this section only assume weak duality. First the following lemma gives a switching identity for energy functionals of dual processes.

Lemma 3.6.1 Let L^q (resp. $\hat{L}^q$) be the energy functionals of X (resp. $\hat{X}$) and $q > 0$. Then for any q-excessive function f and q-coexcessive function $\hat{f}$,

$$L^q(\hat{f}m, f) = \hat{L}^q(fm, \hat{f}). \tag{3.6.1}$$

Proof. First it follows from the duality of resolvents that $(U^q g)\cdot m = (gm)\hat{U}^q$. By property of energy functional, we have $L^q(\hat{f}, U^q g) = \hat{L}^q((U^q g)m, \hat{f})$. Then (3.6.1) follows from the fact that any q-excessive function is the limit of an increasing sequence of q-potentials. □

Now comes the simplest switching identity.

Theorem 3.6.1 (Switching identity I) Let $L \in \mathscr{A}(X)$ and $\hat{K} \in \mathscr{A}(X)$. Then for $f, g \geq 0$,

$$\nu_L(\hat{U}^q_{\hat{K}} f \otimes g) = \hat{\nu}_{\hat{K}}(U^q_L g \otimes f). \tag{3.6.2}$$

Proof. Since $\hat{U}^q_{\hat{K}} f$ is q-coexcessive and $U^q_L g$ is q-excessive, by Lemma 3.6.1 we have

$$L^q(\hat{U}^q_{\hat{K}} f \cdot m, U^q_L g) = \hat{L}^q(U^q_L g \cdot m, \hat{U}^q_{\hat{K}} f).$$

However by (8.7) in [23], $L^q(\hat{U}^q_{\hat{K}} f \cdot m, U^q_L g) = \nu_L(\hat{U}^q_{\hat{K}} f \otimes g)$. That completes the proof. □

Let M be a multiplicative functional of X, which is decreasing, and $\hat{M}$ the dual of M, which is certainly also decreasing. It is well-known that M gives birth to another right Markov process which we call the M-subprocess of X and write as (X, M). For any $L \in \mathscr{A}(X)$, we define the M-potential

$$V^q_L f(x) := \mathsf{P}^x \int_0^\infty e^{-qt} M_{t-} f(X_t) dL_t, \ x \in E. \tag{3.6.3}$$

However V_L^q may also be viewed as the potential operator of M-additive functional $M_- \cdot L := \int M_- dL$. When L is continuous, $M_- \cdot L = M \cdot L$. If $[M]$ is the Stieltjes logarithm of M, then $M_- \cdot d[M] = dM$. In this way, we also have the following nice identity.

Theorem 3.6.2 [Switching identity II] Let $L \in \mathscr{A}(X)$ and $\hat{K} \in \mathscr{A}(\hat{X})$. Then for $f, g \geq 0$,

$$\nu_L(\hat{V}^q_{\hat{K}} f \otimes g) = \hat{\nu}_{\hat{K}}(V^q_L g \otimes f). \tag{3.6.4}$$

Proof. Let L^q_M (resp. $\hat{L}^q_{\hat{M}}$) be the energy functional of the subprocess (X, M) (resp. $(\hat{X}, \hat{M})$). By Lemma 3.6.1, we find

$$L^q_M(\hat{V}^q_{\hat{K}} f \cdot m, V^q_L g) = \hat{L}^q_{\hat{M}}(V^q_L g \cdot m, \hat{V}^q_{\hat{L}} f).$$

By Lemma I.4.5 in [51] or Lemma 3.3.1, it follows that

$$L^q_M(\hat{V}^q_{\hat{K}} f \cdot m, V^q_L g) = \rho^{\hat{V}^q_{\hat{K}} f \cdot m}_{M_- \cdot L}(g),$$

where the right hand side is a Revuz measure of $M_- \cdot L$ taken with respect to the M-excessive measure $\hat{V}^q_{\hat{K}} f \cdot m$. Since $\hat{V}^q_{\hat{K}} f$ is cofinely continuous (it can be decomposed into the difference of two q-coexcessive functions), $t \mapsto \hat{V}^q_{\hat{K}} f(X_{t-})$ is left continuous a.s. Thus by Theorem I.5.5 in [49] or Theorem 3.4.1 (it is easily seen that this theorem is true for m^* replaced with an M-excessive measure), we have

$$\rho^{\hat{V}^q_{\hat{K}} f \cdot m}_{M_- \cdot L}(g) = \nu_{M_- \cdot L}(\hat{V}^q_{\hat{K}} f \otimes g).$$

Finally by Corollary 3.13 in [49] or Corollary 2.2.1, $\nu_{M_- \cdot L} = \nu_L$. That completes the proof. □

Remark. We would like to point out several special cases of identity (3.6.4):

$$(\hat{V}^q_{\hat{K}} f, g) = \hat{\nu}_{\hat{K}}(V^q g \otimes f), \text{ if } dL_t = dt;$$
$$(V^q_L g, f) = \nu_L(\hat{V}^q f \otimes g), \text{ if } d\hat{K}_t = dt;$$
$$(\hat{U}^q_{\hat{K}} f, g) = \hat{\nu}_{\hat{K}}(U^q g \otimes f), \text{ if } dL_t = dt \text{ and } A = 0;$$

$$(U_L^q g, f) = \nu_L(\hat{U}^q f \otimes g), \text{ if } d\hat{K}_t = dt \text{ and } A = 0.$$

The first two were called the generalized Revuz formulas and proved in [51] or §3.4, and the other two were called Revuz formulas and proved in [23]. Finally we would like to point out an application of Theorem 3.6.2.

Corollary 3.6.1 Assume that X and $\hat{X}$ are dual with respect to m and their corresponding PCAF's A and $\hat{A}$ are also dual. Then their time changes X_τ and $\hat{X}_{\hat{\tau}}$ are dual with respect to $\tilde{m}$ which is the Revuz measure of A (or $\hat{A}$).

Proof. We need to show that for $f, g \geq 0$

$$\langle \tilde{U}^\alpha f, g\rangle_{L^2(\tilde{m})} = \langle f, \hat{\tilde{U}}^\alpha g\rangle_{L^2(\tilde{m})},$$

namely

$$\langle \mathsf{E}^\cdot \int_0^\infty e^{-\alpha A_t} f(X_t) dA_t, g\rangle = \langle f, \hat{\mathsf{E}}^\cdot \int_0^\infty e^{-\alpha \hat{A}_t} f(\hat{X}_t) d\hat{A}_t\rangle.$$

This follows directly from Theorem 3.6.2. □

Chapter 4

Feynman-Kac Formula of Dirichlet forms

Though the work on symmetric Markov processes has been completed in [50], the similar result for non-symmetric processes is not just trivial generalization because the method employed in [10] or [50] can not be used in non-symmetric case. Given a nearly symmetric process, i.e., one that satisfies the 'sector condition' of Silverstein [41], [42], which would guarantee a nice Dirichlet form associated, and any of its multiplicative functionals, there are two questions to be raised: (1) whether or not is the subprocess still nearly symmetric? (2) if it is, how to describe the Dirichlet form of the subprocess? Our main result in this part is to prove that the subprocess is also nearly symmetric and a similar Feynman-Kac formula holds.

All notations and terminologies without description are inherited from chapter 3, which we won't restate here. Only the notion of Dirichlet forms will be given below. After collecting in §4.1 some definitions and standard facts concerning Dirichlet forms and multiplicative functionals, which can be found in [30] and [10], we are going to formulate our main result in §4.2 where the involved MF is assumed to be non-vanishing and in §4.3 for general MF's.

4.1 Basic notion of Dirichlet forms

Since X and $\hat{X}$ are right processes in duality with respect to a σ-finite measure m on $(E, \mathscr{E})$, (P_t) is a strongly continuous semigroup of contractions on $L^2(m)$. Let A denote the strong $L^2(m)$-infinitesimal generator of (P_t), with domain $D(A)$. Then $D(A)$ is dense in $L^2(m)$, A is closed, $-A$ is positive: $(f, -Af) \geq 0$ for all $f \in D(A)$. The sector condition is this

(4.1.1) $$(f, -Ag) \leq K \cdot (f, -Af)^{\frac{1}{2}}(g, -Ag)^{\frac{1}{2}}, \quad f, g \in D(A)$$

for some constant $K > 0$. Of course (4.1.1) is always valid in the symmetric case. The important consequences of (4.1.1) are that every semipolar set is m-polar and X and $\hat{X}$ are not only Borel right processes but also m-special standard ones. Thus we can make free use of the results in Chapter 3.

Let $(\mathscr{E}, \mathscr{F})$ be a bilinear form on $L^2(m)$. Define for $u, v \in \mathscr{F}$,

(4.1.2) $$\begin{aligned} \bar{\mathscr{E}}(u, v) &:= \frac{1}{2}[\mathscr{E}(u, v) + \mathscr{E}(v, u)]; \\ \mathscr{E}_\alpha(u, v) &:= \mathscr{E}(u, v) + \alpha(u, v); \\ [\mathscr{E}_\alpha](u) &:= \mathscr{E}_\alpha(u, u)^{\frac{1}{2}}. \end{aligned}$$

$[\mathscr{E}_\alpha](u)$ is also called the $\mathscr{E}_\alpha$-norm of u.

Definition 4.1.1 A pair $(\mathscr{E}, \mathscr{F})$ is called a coercive closed form on $L^2(m)$ if $\mathscr{F}$ is a dense linear subspace of $L^2(m)$ and $\mathscr{E} : \mathscr{F} \times \mathscr{F} \longrightarrow \mathbf{R}$ is a bilinear form such that the following two conditions hold.

(i) Its symmetric part $(\bar{\mathscr{E}}, \mathscr{F})$ is a symmetric closed form on $L^2(m)$; that is, $\bar{\mathscr{E}} : \mathscr{F} \times \mathscr{F} \longrightarrow \mathbf{R}$ is a positive definite bilinear form and $\mathscr{F}$ is complete with respect to $\mathscr{E}_1$-norm.

(ii) $(\mathscr{E}, \mathscr{F})$ satisfies the sector condition

(4.1.3) $$|\mathscr{E}(u, v)| \leq K \cdot \mathscr{E}(u, u)^{\frac{1}{2}} \cdot \mathscr{E}(v, v)^{\frac{1}{2}}$$

constant $K > 0$.

The sector condition (4.1.3) is equivalent to

(4.1.4) $$|\operatorname{Im} \mathscr{E}(u, \bar{u})| \leq K \cdot \operatorname{Re} \mathscr{E}(u, \bar{u})$$

for $u \in \mathscr{F} + i\mathscr{F}$, where $\bar{u}$ is the conjugate of u.

Proposition 4.1.1 Let $(\mathscr{E}, \mathscr{F})$ be a coercive closed form on $L^2(m)$. Then there exist unique strongly continuous contraction resolvents (G_α), $(\hat{G}_\alpha)$ on $L^2(m)$ such that $G_\alpha(L^2(m))$, $\hat{G}_\alpha(L^2(m)) \subset \mathscr{F}$ and $\mathscr{E}_\alpha(u, G_\alpha f) = (u, f) = \mathscr{E}_\alpha(\hat{G}_\alpha f, u)$ for all $f \in L^2(m)$, $u \in \mathscr{F}$ and $\alpha > 0$.

In this case (G_α) and $(\hat{G}_\alpha)$ are called the resolvents associated with $(\mathscr{E}, \mathscr{F})$.

Definition 4.1.2 A coercive closed form $(\mathscr{E}, \mathscr{F})$ on $L^2(m)$ is called a Dirichlet form if for all $u \in \mathscr{F}$, $u^+ \wedge 1 \in \mathscr{F}$ and

$$\mathscr{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0, \quad \mathscr{E}(u - u^+ \wedge 1, u + u^+ \wedge 1) \geq 0.$$

Proposition 4.1.2 Suppose $(\mathscr{E}, \mathscr{F})$ is a coercive closed form on $L^2(m)$ with corresponding resolvents (G_α) and $(\hat{G}_\alpha)$. Then the following two conditions are equivalent.
(1) $(\mathscr{E}, \mathscr{F})$ is a Dirchlet form on $L^2(m)$.
(2) (G_α) and $(\hat{G}_\alpha)$ are sub-Markovian.

Now beginning with the process X, we define bilinear forms $\mathscr{E}_\alpha$, $\alpha \geq 0$, by

$$\mathscr{E}_\alpha(f, g) = (f, -Ag) + \alpha(f, g), \quad f, g \in D(A). \tag{4.1.5}$$

We write $\mathscr{E}$ for $\mathscr{E}_0$ and $[\mathscr{E}_\alpha](f) = \mathscr{E}_\alpha(f, f)^{\frac{1}{2}}$ for $f \in D(A)$. Clearly $[\mathscr{E}_\alpha]$ is a norm on $D(A)$ for $\alpha > 0$. Let $\mathscr{F}$ denote the completion of $D(A)$ relative to $[\mathscr{E}_1]$. Owing to (4.1.1), $\mathscr{E}$ extends uniquely to a bilinear form (also denoted $\mathscr{E}$) on $\mathscr{F}$. The pair $(\mathscr{E}, \mathscr{F})$ is the Dirichlet form associated with X and

$$\mathscr{F} = \{u \in L^2(m) : \sup_q q(u, u - qU^{q+1}u) < \infty\}. \tag{4.1.6}$$

Let J be the canonical measure of X relative to m. According to [30], the Dirichlet form $\mathscr{E}$, which is associated with a pair of m-special standard processes, is quasi-regular. By the 'transfer method' developed in Chapter VI of [30], the Fukushima's decomposition holds. Therefore we have the following Beurling-Deny's decomposition for $\mathscr{E}$ on the diagonal

(4.1.7) $$\mathscr{E}(u,u) = \mathscr{E}^c(u,u) + \frac{1}{2}\int [u(x)-u(y)]^2 J(dx,dy)$$

for $u \in \mathscr{F}$, where $\mathscr{E}^c$ is a Dirichlet form corresponding the continuous part of X.

Proposition 4.1.3 Let $\{f_n\}$ be a sequence in $\mathscr{F}$ such that $\sup_n [\mathscr{E}_1](f_n) < \infty$ and $f_n \to f$ a.e. Then $f \in \mathscr{F}$ and $f_n \to f$ weakly in $\mathscr{F}$, i.e., $\mathscr{E}_1(f_n, g) \to \mathscr{E}_1(f,g)$ for each $g \in \mathscr{F}$.

4.2 Feynman-Kac formula: the non-vanishing case

Unless otherwise stated we assume in this section that $M = (M_t)$ is a multiplicative functional of X which never vanishes before ζ. There exists its dual multiplicative functional $\hat{M}$, non-vanishing, such that (X, M) and $(\hat{X}, \hat{M})$ are in weak duality relative to m. Denote their semigroups and resolvents by (Q_t), (V^q) and $(\hat{Q}_t)$, $(\hat{V}^q)$, respectively. The Stieltjes logarithm of M is defined by

(4.2.1) $$[M]_t := \int_0^t \frac{d(-M_s)}{M_{s-}}.$$

Then $[M]$ is an additive functional of X and we have the following identities

(4.2.2) $$U^q = V^q + U^q_{[M]} V^q; \quad U^p_{[M]} = U^{q+p}_{[M]} + qU^{q+p} U^p_{[M]}.$$

Since $(1-M_t) \in \mathrm{AF}(M)$, the bivariate Revuz measure of $(1-M_t)$ relative to m makes sense and is denoted by ν, which is also called the bivariate Revuz measure of M. Let J be the canonical measure of X relative to m. We know from Theorem 3.3.1 that $\nu \leq J$ off the diagonal of $E \times E$. Denote by ρ and λ the marginal measures of ν; that is, for any $f \in \mathscr{E}_+$

$$\rho(f) = \nu(1 \otimes f), \quad \lambda(f) = \nu(f \otimes 1).$$

Finally we list some facts, which may be either easy to see or be found in Chapter 3, for handy reference.

(4.2.3) $\nu = \nu_{[M]}$, where $\nu_{[M]}$ is the bivariate Revuz measure of $[M]$.

(4.2.4) The Revuz formula $(f, U^{\alpha}_{[M]}g) = \nu_{[M]}(\hat{U}^{\alpha} f \otimes g)$ and generalized Revuz formula $(f, U^{\alpha}_{M}g) = \nu(\hat{V}^{\alpha} f \otimes g)$ for $f, g \in \mathscr{E}_+$.

(4.2.5) By Hölder's inequality, we have $|\nu(f \otimes g)| \leq \lambda(f^2)^{\frac{1}{2}} \cdot \rho(g^2)^{\frac{1}{2}}$.

(4.2.6) The dual statements of all above are true and furthermore ν and $\hat{\nu}$ are dual; that is, $\nu(dx, dy) = \hat{\nu}(dy, dx)$.

Let $(B, D(B))$ be the strong $L^2(m)$-infinitesimal generator of the semigroup (Q_t) of the M-subprocess. It is known that $D(B) = V^1(L^2(m)) = V^{\alpha}(L^2(m))$ for $\alpha > 0$.

Lemma 4.2.1 (a) $D(B) \subset \mathscr{F} \cap L^2(\rho) \cap L^2(\lambda)$. (b) The following formula holds

$$(u, g) = \mathscr{E}_{\alpha}(u, V^{\alpha} g) + \nu(u \otimes V^{\alpha} g) \tag{4.2.7}$$

for any $u \in \mathscr{F}$, $g \in L^2(m)$, $u, g \geq 0$ and $\alpha > 0$.

Proof. Let $L^2_+(m) := \{f \in L^2(m) : \ f \geq 0\}$. First of all we will show $V^1(L^2_+(m)) \subset \mathscr{F}$. In fact for $u = V^1 f$ with $f \in L^2_+(m)$, using (4.2.2) we have

$$\begin{aligned} q(u, u - qU^{q+1}u) &= q(u, U^1 f - qU^{q+1}U^1 f - U^1_{[M]}u + qU^{q+1}U^1_{[M]}u) \\ &= q(u, U^{q+1} f - U^{q+1}_{[M]}u) \\ &\leq (u, qU^{q+1} f). \end{aligned}$$

Since $(u, qU^{q+1} f) \longrightarrow (u, f)$ as $q \to \infty$, $\sup_q q(u, u - qU^{q+1}u) < \infty$ and then $u \in \mathscr{F}$ by (4.1.6).

Now for any u, g and α as in (b), we have by (4.2.2)

$$(u, g) = \mathscr{E}_{\alpha}(u, U^{\alpha} g) = \mathscr{E}_{\alpha}(u, V^{\alpha} g) + \mathscr{E}_{\alpha}(u, U^{\alpha}_{[M]} V^{\alpha} g).$$

Then using (4.2.4)

$$\mathscr{E}_{\alpha}(u, U^{\alpha}_{[M]} V^{\alpha} g) = \lim_{\alpha} q(u, U^{\alpha}_{[M]} V^{\alpha} g - qU^{q+\alpha} U^{\alpha}_{[M]} V^{\alpha} g)$$

$$= \lim_q (u, qU_{[M]}^{q+\alpha} V^\alpha g)$$
$$= \lim_q \nu_{[M]}(q\hat{U}^{q+\alpha} u \otimes V^\alpha g).$$

Since $u \in \mathscr{F}$, $q\hat{U}^{q+\alpha}u \longrightarrow u$ q.e. as $q \longrightarrow \infty$ and by (4.2.3) and the generalized Revuz formula

$$\nu_{[M]}(1 \otimes V^\alpha g) = \nu(1 \otimes V^\alpha g) = \hat{\nu}(V^\alpha g \otimes 1) = (g, \hat{U}^q_{\hat{M}} 1) \leq m(g).$$

Thus if, in addition, u is bounded and $g \in L^1(m)$, then $\{q\hat{U}^{q+\alpha}u\}$ is uniformly bounded and $\nu(dx \otimes V^\alpha g)$ is a finite measure, and hence (4.2.7) holds by the dominated convergence theorem. Now for $g \in L^1 \cap L^2(m)$ and any $u \in \mathscr{F}$, we write $u_n = u \wedge n$ and then $u_n \uparrow u$ and in $\mathscr{F}$. Since u_n is bounded, $(u_n, g) = \mathscr{E}_\alpha(u_n, V^\alpha g) + \nu(u_n \otimes V^\alpha g)$. Passing to limit, by the monotone convergence theorem, (3.8) holds in this case. Finally for any $g \in L^2(m)$, we can pick h strictly positive on E and $m(h) < \infty$. Let $g_n = g \wedge (nh)$. Then $g_n \in L^1 \cap L^2(m)$, $g_n \uparrow g$ pointwisely and

$$(u, g_n) = \mathscr{E}_\alpha(u, V^\alpha g_n) + \nu(u \otimes V^\alpha g_n).$$

Then $V^\alpha g_n \uparrow V^\alpha g$ by the monotone convergence theorem. Also we have

$$\begin{aligned}
\mathscr{E}_\alpha(V^\alpha g_n, V^\alpha g_n) &= \lim_q q(V^\alpha g_n, V^\alpha g_n - qU^{q+\alpha} V^\alpha g_n) \\
&\leq \lim_q (V^\alpha g_n, qU^{q+\alpha} g_n) \\
&= (V^\alpha g_n, g_n) \\
&\leq (V^\alpha g, g).
\end{aligned}$$

Thus by (4.1.3), $V^\alpha g_n \longrightarrow V^\alpha g$ weakly in $\mathscr{F}$ and (4.2.7) holds for positive $u \in \mathscr{F}$, $g \in L^2(m)$ and $\alpha > 0$ by another use of the monotone convergence theorem.

Now for $f \in L^2_+(m)$, $u = V^\alpha f \in \mathscr{F}$ and by (4.2.7) and (4.1.7)

$$\begin{aligned}
(V^\alpha f, f) &= \mathscr{E}_\alpha(u, u) + \nu(u \otimes u) \\
&= \mathscr{E}^c_\alpha(u, u) + \frac{1}{2}\int [(u(x) - u(y)]^2 (J - \nu)(dx, dy) + \frac{1}{2}(\rho(u^2) + \lambda(u^2)).
\end{aligned}$$

Thus $\rho(u^2) + \lambda(u^2) < \infty$, i.e., $u \in L^2(\rho) \cap L^2(\lambda)$ and

$$D(B) = V^1(L^2_+(m)) - V^1(L^2_+(m)) \subset \mathscr{F} \cap L^2(\rho) \cap L^2(\lambda).$$

That completes the proof. □

We now define

$$\begin{aligned} \mathscr{F}' &= \mathscr{F} \cap L^2(\rho) \cap L^2(\lambda); \\ \mathscr{E}'(u,v) &= \mathscr{E}(u,v) + \nu(u \otimes v), \quad u, v \in \mathscr{F}'. \end{aligned} \tag{4.2.8}$$

By (4.2.5), $(\mathscr{E}', \mathscr{F}')$ is a well-defined bilinear form on $L^2(m)$.

Theorem 4.2.1 The bilinear form $(\mathscr{E}', \mathscr{F}')$ is a Dirichlet form on $L^2(m)$ associated with the M-subprocess of X. In other words the M-subprocess of a nearly symmetric Markov process with the Dirichlet form $(\mathscr{E}, \mathscr{F})$ is also nearly symmetric and its associated Dirichlet form $(\mathscr{E}', \mathscr{F}')$ is given in (4.2.8).

Proof. First we will show that $(\mathscr{E}', \mathscr{F}')$ is a coercive closed form on $L^2(m)$. Clearly $\mathscr{F}'$ is dense in $L^2(m)$ since $D(B) \subset \mathscr{F}'$ as in Lemma 4.2.1. The decomposition of $\mathscr{E}'$ on diagonal is

(4.2.9)
$$\begin{aligned} \mathscr{E}'(u,u) &= \mathscr{E}(u,u) + \nu(u \otimes u) \\ &= \mathscr{E}^c(u,u) + \frac{1}{2}\int [u(x) - u(y)]^2 J'(dx,dy) + \frac{1}{2}(\rho(u^2) + \lambda(u^2)), \end{aligned}$$

where $J' = J - \nu|_{E\times E-d}$, the canonical measure of the M-subprocess. Hence $\mathscr{E}'(u,u) \geq 0$. Now we need to check that $\mathscr{F}'$ is $[\mathscr{E}'_1]$-complete. Let $\{u_n\}$ be any $[\mathscr{E}'_1]$-Cauchy sequence in $\mathscr{F}'$, i.e., $[\mathscr{E}'_1](u_n - u_m) \longrightarrow 0$. Then by (4.2.9) $\{u_n\}$ is a $L^2(\rho)$ and $L^2(\lambda)$-Cauchy sequence. Thus by (4.2.6), $\nu((u_n - u_m) \otimes (u_n - u_m)) \longrightarrow 0$ and $[\mathscr{E}_1](u_n - u_m) \longrightarrow 0$; that is, $\{u_n\}$ is also an $[\mathscr{E}_1]$-Cauchy sequence in $\mathscr{F}$. There exists $u \in \mathscr{F}$ such that $u_n \longrightarrow u$ strongly in $\mathscr{F}$. Then $u_n \longrightarrow u$ q.e. (at least for a subsequence), and consequently $u_n \longrightarrow u$ a.e. ρ and λ. Hence u coincides with the L^2-limit of $\{u_n\}$ in $L^2(\rho)$ and $L^2(\lambda)$. Therefore $u \in \mathscr{F} \cap L^2(\rho) \cap L^2(\lambda) = \mathscr{F}'$ and $[\mathscr{E}'_1](u_n - u) \longrightarrow 0$.

Next we will prove that $(\mathscr{E}', \mathscr{F}')$ satisfies the sector condition (4.1.4). Let $u = f + ig$ with $f, g \in \mathscr{F}'$. We need to show that

$$|\operatorname{Im} \mathscr{E}'(u, \bar{u})| \le K \cdot \operatorname{Re} \mathscr{E}'(u, \bar{u}) \tag{4.2.10}$$

for a constant $K > 0$. In fact $\mathscr{E}'(u, \bar{u}) = \mathscr{E}(u, \bar{u}) + \nu(u \otimes \bar{u})$ and

$$\begin{aligned}
\mathscr{E}(u, \bar{u}) &= \mathscr{E}(f, f) + \mathscr{E}(g, g) + i \operatorname{Im} \mathscr{E}(u, \bar{u}) \\
&= \operatorname{Re} \mathscr{E}^c(u, \bar{u}) + i \operatorname{Im} \mathscr{E}(u, \bar{u}) \\
&+ \frac{1}{2} \int ([f(x) - f(y)]^2 + [g(x) - g(y)]^2) J(dx, dy) \\
&= \operatorname{Re} \mathscr{E}^c(u, \bar{u}) + i \operatorname{Im} \mathscr{E}(u, \bar{u}) \\
&+ \int (\frac{1}{2}(|u(x)|^2 + |u(y)|^2) - f(x)f(y) - g(x)g(y)) J(dx, dy); \\
\nu(u \otimes \bar{u}) &= \nu(f \otimes f) + \nu(g \otimes g) + i\nu(g \otimes f - f \otimes g) \\
&= - \int (\frac{1}{2}(|u(x)|^2 + |u(y)|^2) - f(x)f(y) - g(x)g(y)) \nu(dx, dy) \\
&+ \frac{1}{2}(\rho(|u|^2) + \lambda(|u|^2)) + i\nu(g \otimes f - f \otimes g).
\end{aligned}$$

Define

$$\begin{aligned}
d &:= \frac{1}{2}(\rho(|u|^2) + \lambda(|u|^2)) < \infty; \\
\nu_o &:= \frac{1}{2}(|u|^2 \otimes 1 + 1 \otimes |u|^2) \cdot \nu; \\
J_o &:= \frac{1}{2}(|u|^2 \otimes 1 + 1 \otimes |u|^2) \cdot J; \\
\mathrm{SIN}[u](x, y) &:= \frac{2(g(x)f(y) - f(x)g(y))}{|u(x)|^2 + |u(y)|^2}; \\
\mathrm{COS}[u](x, y) &:= \frac{2(f(x)f(y) + g(x)g(y))}{|u(x)|^2 + |u(y)|^2}.
\end{aligned}$$

Then we have

(4.2.11)

$$\begin{aligned}
\mathscr{E}(u, \bar{u}) &= \operatorname{Re} \mathscr{E}^c(u, \bar{u}) + J_o(1 - \mathrm{COS}[u]) + i \operatorname{Im} \mathscr{E}(u, \bar{u}); \\
\mathscr{E}'(u, \bar{u}) &= \operatorname{Re} \mathscr{E}(u, \bar{u}) + i \operatorname{Im} \mathscr{E}(u, \bar{u}) + \operatorname{Re} \nu(u \otimes \bar{u}) + i \operatorname{Im} \nu(u \otimes \bar{u}) \\
&= \operatorname{Re} \mathscr{E}^c(u, \bar{u}) + J_o(1 - \mathrm{COS}[u]) - \nu_o(1 - \mathrm{COS}[u]) + d \\
&\quad + i \operatorname{Im} \mathscr{E}(u, \bar{u}) + i\nu_o(\mathrm{SIN}[u]),
\end{aligned}$$

and clearly $|\,\mathrm{COS}[u]| \le 1$, $|\,\mathrm{SIN}[u]| \le 1$ and $\nu_o(1) = d$. Now let

$$\begin{aligned} G(u) &:= \mathscr{E}(u,\bar{u}) - \nu_o(1 - \mathrm{COS}[u]) + \frac{d}{2}; \\ H(u) &:= i\nu_o(\mathrm{SIN}[u]) + \frac{d}{2}. \end{aligned} \tag{4.2.12}$$

Then $\mathscr{E}'(u,\bar{u}) = G(u) + H(u)$ and

$$\begin{aligned} \operatorname{Im} G(u) &= \operatorname{Im} \mathscr{E}(u,\bar{u}); \\ \operatorname{Re} G(u) &= \operatorname{Re} \mathscr{E}(u,\bar{u}) - \nu_o(1 - \mathrm{COS}[u]) + \frac{d}{2} \\ &= \operatorname{Re} \mathscr{E}^c(u,\bar{u}) + (J_o - \nu_o)(1 - \mathrm{COS}[u]) + \frac{d}{2}. \end{aligned}$$

It is easy to see that

$$|\operatorname{Im} H(u)| \le \nu(|\,\mathrm{SIN}[u]|) \le d = 2 \operatorname{Re} H(u).$$

Thus we need only to check that $|\operatorname{Im} G(u)| \le K \cdot \operatorname{Re} G(u)$ for a constant K. Since $\mathscr{E}$ satisfies the sector condition, it suffices to show that $\operatorname{Re} \mathscr{E}(u,\bar{u}) \le K' \cdot \operatorname{Re} G(u)$ for a constant K'.

Denote

$$\mathscr{U} = \{u \in \mathscr{F}' + i\mathscr{F}' :\ J_o(1 - \mathrm{COS}[u]) \ge \nu_o(1 - \mathrm{COS}[u]) + 2d\}. \tag{4.2.13}$$

Since $0 \le \nu_o(1 - \mathrm{COS}[u]) \le 2d$, $\nu_o(1 - \mathrm{COS}[u]) \le \frac{1}{2} J_o(1 - \mathrm{COS}[u])$ for $u \in \mathscr{U}$ and then

$$2(J_o - \nu_o)(1 - \mathrm{COS}[u]) \ge J_o(1 - \mathrm{COS}[u]). \tag{4.2.14}$$

Hence for $u \in \mathscr{U}$, we have

$$\begin{aligned} \operatorname{Re} \mathscr{E}(u,\bar{u}) &\le \operatorname{Re} \mathscr{E}^c(u,\bar{u}) + J_o(1 - \mathrm{COS}[u]) \\ &\le \operatorname{Re} \mathscr{E}^c(u,\bar{u}) + 2(J_o - \nu_o)(1 - \mathrm{COS}[u]) \\ &\le 2 \operatorname{Re} G(u). \end{aligned}$$

On the other hand, if $u \notin \mathscr{U}$,

$$\operatorname{Re} \mathscr{E}(u,\bar{u}) \le \operatorname{Re} \mathscr{E}^c(u,\bar{u}) + \nu_o(1 - \mathrm{COS}[u]) + 2d$$

$$\begin{aligned}
&\leq \operatorname{Re}\mathscr{E}^c(u,\bar{u}) + 4d \\
&\leq 8(\operatorname{Re}\mathscr{E}^c(u,\bar{u}) + \frac{d}{2}) \\
&\leq 8\operatorname{Re}G(u).
\end{aligned}$$

Therefore for any $u \in \mathscr{F}' + i\mathscr{F}'$, $\operatorname{Re}\mathscr{E}(u,\bar{u}) \leq 8\operatorname{Re}G(u)$; that is, $(\mathscr{E}', \mathscr{F}')$ satisfies the sector condition and it is a coercive closed form on $L^2(m)$. By Lemma 4.2.1, it is clear that

$$\mathscr{E}'_\alpha(u, V^\alpha f) = (u, f) \tag{4.2.15}$$

for all $u \in \mathscr{F}'$, $f \in L^2(m)$ and $\alpha > 0$. The dual assertion can be shown similarly. Hence by the uniqueness in Proposition 4.1.1, (V^q) and $(\hat{V}^q)$ are the resolvents associated with $(\mathscr{E}', \mathscr{F}')$. Since (V^q) and $(\hat{V}^q)$ are sub-Markovian, it follows form Proposition 4.1.2 that $(\mathscr{E}', \mathscr{F}')$ is a Dirichlet form on $L^2(m)$, which is associated with the M-subprocess of X. □

4.3 Feynman-Kac formula: General case

In this section we are going to generalize Theorem 4.2.1 to general multiplicative functionals.

Theorem 4.3.1 Let M be an exact multiplicative functional of X and $m^* := 1_{E_M} \cdot m$. Then the subprocess (X, M) is a nearly m^*-symmetric Markov process on $(E_M, \mathscr{E} \cap E_M)$ and its associated Dirichlet form $(\mathscr{E}', \mathscr{F}')$ is given by

$$\begin{aligned}
&\mathscr{F}' = (\mathscr{F})_{E_M} \cap L^2(\rho_M + \lambda_M); \\
&\mathscr{E}'(u,v) = \mathscr{E}(u,v) + \nu_M(u \otimes v), \quad u, v \in \mathscr{F}',
\end{aligned} \tag{4.3.1}$$

where

$$(\mathscr{F})_{E_M} := \{u \in \mathscr{F} : \ u = 0 \text{ q.e. on } E_M^c\}$$

the restricted Dirichlet form of $\mathscr{E}$ on E_M.

Proof. The transformation by M can be completed in three steps: killing X by a hitting time $T_{E_M^c}$, denote the resulting one by X', killing X' by a positive terminal time S_M of X', denote the resulting one by X'', killing X'' by a non-vanishing MF M of X''. Because of the works on non-vanishing MF's in Theorem 4.2.1 and on restricted Dirichlet forms in [30], and a connection formula (3.3.10), it suffices to deal with the case of positive terminal times, i.e., $M = 1_{[0,T[}$, where T is a terminal time and $T > 0$ a.s. Then there exists $B \subset E \times E - D$ such that P^m a.s.

$$T = J_B := \inf\{t > 0 : \ (X_{t-}, X_t) \in B\}. \tag{4.3.2}$$

We will finishing the proof by two steps. First assume that $B \subset \{(x, y) : d(x, y) > c\}$ for some $c > 0$ where d is a metric on E compatible with the original topology on E. Define $T^{(1)} := T$, $T^{(n+1)} := T \circ \theta_{T^{(n)}} + T^{(n)}$. Since X is right continuous,

$$\sum_{s \le t} 1_B(X_{s-}, X_s) = \sum_n 1_{\{T^{(n)} \le t\}} < \infty \tag{4.3.3}$$

for any $t < \zeta$. Hence for any $0 \le \delta < 1$, $M_t^\delta := \prod_{s \le t}[1 - \delta 1_B(X_{s-}, X_s)]$ is a non-vanishing multiplicative functional of X. Let $\delta_n := \frac{n}{n+1}$ for $n \ge 0$, $M_t^n := M_t^{\delta_n}$ and (V_n^q) be the resolvent of the subprocess (X, M^n). Clearly $V_n^q \downarrow V^q f$ pointwisely for $f \ge 0$ and $\nu_{M^n} = \delta_n \nu_M \uparrow \nu_M$. Define

$$\begin{aligned} \mathscr{F}' &:= \mathscr{F} \cap L^2(\rho_M + \lambda_M); \\ \mathscr{E}'(u, v) &:= \mathscr{E}(u, v) + \nu(M(u \otimes v), \quad u, v \in \mathscr{F}'. \end{aligned} \tag{4.3.4}$$

We need only to show that $(\mathscr{E}', \mathscr{F}')$ is the Dirichlet form and associated with (X, T). Following the proof of Theorem 4.2.1, it suffices to check the following claims.

$$V^1 f \in \mathscr{F} \text{ for any } f \in L^2_+(m); \tag{4.3.5}$$

$$\mathscr{E}'_1(u, V^1 f) = (u, f) \text{ for any } u \in p\mathscr{F},\ f \in L^2_+(m). \tag{4.3.6}$$

Since M^n never vanishes, we know that $V_n^1 f \in \mathscr{F}$ and

$$(u, f) = \mathscr{E}_1(u, V_n^1 f) + \nu_{M^n}(u \otimes V_n^1 f). \tag{4.3.7}$$

Since $\mathscr{E}_1(V_n^1 f, V_n^1 f) \le (V_n^1 f, f) \le (U^1 f, f) < \infty$ and $V_n^1 f \downarrow V^1 f$ pointwisely, $V^1 f \in \mathscr{F}$ and $V_n^1 f \longrightarrow V^1 f$ weakly in $(\mathscr{F}, \mathscr{E}_1)$. We also have

$$\nu_M(u \otimes V_1^1 f) = 2\nu_{M^1}(u \otimes V_1^1 f) < \infty.$$

Hence take n to infinity in (4.3.7) and the dominated convergence theorem gives

$$\begin{aligned}(u, f) &= \mathscr{E}_1(u, V^1 f) + \lim_n \nu_M(u \otimes V_n^1 f)\\ &= \mathscr{E}_1(u, V^1 f) + \nu_M(u \otimes V^1 f)\\ &= \mathscr{E}_1'(u, V^1 f).\end{aligned}$$

That completes the proof of the first part (4.3.5). Next we need to check (4.3.6) for general $B \subset E \times E - D$. Set

$$B_n := B \cap \{(x, y) : \ d(x, y) > \frac{1}{n}\} \ \text{ and } \ T_n := J_{B_n}.$$

Since $T > 0$ a.s., $\{T_n\}$ **well** converges decreasingly to T in the sense that for any $\omega \in \Omega$ there exists $N = N(\omega)$ such that $T_n(\omega) = T(\omega)$ for all $n > N$. In fact by Theorem 3.2.1 $(X_{T-}, X_T) \in B$. Thus for such an ω, there exists N such that $(X_{T-}, X_T) \in B_N$ and this N guarrantees $T_n(\omega) = T(\omega)$ for $n > N$.

Denote by (V_n^q) the resolvent of (X, T_n). Then

$$V_n^q f(x) = \mathsf{P}^x \int_{]0,T_n[} e^{-qt} f(X_t) dt$$

decreases to

$$\mathsf{P}^x \int_{]0,T[} e^{-qt} f(X_t) dt = V^q f(x).$$

we can surely apply the result above to T_n, and have

$$(u, f) = \mathscr{E}_1(u, V_n^1 f) + \nu_{T_n}(u \otimes V_n^1 f).$$

The similar reasoning gives $V^1 f \in \mathscr{F}$ and $V_n^1 f \longrightarrow V^1 f$ in $(\mathscr{F}, \mathscr{E}_1)$. On the other hand let $\hat{B} = \{(x, y) : \ (y, x) \in B\}$, $\hat{T} = \hat{J}_{\hat{B}} := \inf\{t > 0 : \ (\hat{X}_{t-}, \hat{X}_t) \in$

$\hat{B}\}$ and $\hat{T}_n := \hat{J}_{\hat{B}_n}$. Then $\hat{T}$ (resp. $\hat{T}_n$) is dual to T (resp. T_n), $\hat{\nu}_{\hat{T}}$ (resp. $\hat{\nu}_{\hat{T}_n}$) is dual to and $\hat{T}_n$ well converges to $\hat{T}$. Using the dual form of generalized Revuz formula (3.4.5) and (4.2.6) we have

$$\nu_{T_n}(u \otimes V_n^1 f) = \hat{\nu}_{\hat{T}_n}(V_n^1 f \otimes u) = (f, \hat{P}^1_{\hat{T}_n} u).$$

Now $\hat{P}^1_{\hat{T}_n} u(x) = \hat{\mathsf{P}}^x e^{-\hat{T}_n} u(\hat{X}_{\hat{T}_n})$. By the well convergence of $\{T_n\}$,

$$\hat{P}^1_{\hat{T}_n} u(x) \longrightarrow \hat{\mathsf{P}}^x \left(e^{-\hat{T}} u(\hat{X}_{\hat{T}}) \right) = \hat{P}^1_{\hat{T}} u(x),$$

i.e., $\hat{P}^1_{\hat{T}_n} u \longrightarrow \hat{P}^1_{\hat{T}} u$ pointwisely. Hence for $f \in L^1_+(m)$ and bounded u,

$$(f, \hat{P}^1_{\hat{T}_n} u) \longrightarrow (f, \hat{P}^1_{\hat{T}} u).$$

Using the standard techniques, we have this convergence for $u \in p\mathscr{F}$ and $f \in L^2_+(m)$. Then applying the dual form of (3.4.5) again

$$\begin{aligned}(u, f) &= \mathscr{E}_1(u, V^1 f) + (f, \hat{P}^1_{\hat{T}} u) \\ &= \mathscr{E}_1(u, V^1 f) + \hat{\nu}_{\hat{T}}(V^1 f \otimes u) \\ &= \mathscr{E}_1(u, V^1 f) + \nu_T(u \otimes V^1 f) \\ &= \mathscr{E}'_1(u, V^1 f).\end{aligned}$$

That completes the proof. □

4.4 h-transforms and drift transforms

In this paper we are going to give a formula characterizing Revuz measures under h-transform. Assume that X is a right Markov process with state space $(E, \mathscr{E})$ which is metrizable, constructed on the canonical space Ω of right continuous paths, and (P_t) and (U_q) are the semigroup and resolvent of X. Let h be an excessive function and let $E_h := \{0 < h < \infty\}$. Define kernels P_t^h by

$$P_t^h(x, dy) = \frac{1}{h(x)} P_t(x, dy) h(y), \; x \in E_h; \tag{4.4.1}$$

$$= \epsilon_x(dy), \ x \in E - E_h.$$

Then it is well known and easy to check that (P_t^h) is a sub-Markovian semigroup on E. It is also known (see, e.g. [37], [24]) that there exist probabilities $\mathsf{P}^{x/h}$ on Ω for $x \in E$ such that $X^h := (X_t, \mathsf{P}^{x/h})$ is a right process with state space $(E, \mathscr{B})$ and semigroup (P_t^h). Clearly $X = X^1$. We call X^h the h-transform of X (by h) and denote its resolvent by (U_q^h). We make the assumption that $E_h = E$ in this paper just for convenience. The notations $\mathscr{S}^q(h)$ and $\text{Exc}^q(h)$, $q \geq 0$, are used for the classes of functions and measures respectively excessive relative to the semigroup $(e^{-qt}P_t^h)$ (or called q, h-excessive). Particularly $\mathscr{S}^q := \mathscr{S}^q(1)$ and $\text{Exc}^q := \text{Exc}^q(1)$. (By convention q will be erased if $q = 0$.) Recall that $v \in \mathscr{S}(h)$ if and only if $vh \in \mathscr{S}$ while $\xi \in \text{Exc}$ if and only if $h\xi \in \text{Exc}(h)$. Also known as in [24] if L_h denotes the energy functional of X^h, $L_h(h\xi, v) = L(\xi, hv)$ where $\xi \in \text{Exc}$, $v \in \mathscr{S}(h)$ and $L := L_1$ the energy functional of X.

As a convention for notations, the 'p' and 'b' before a class of functions stand for 'nonnegative' and 'bounded' respectively. For any measure μ and function f, $\mu(f)$ is a shorthand for the integral $\int f d\mu$. For two functions f, g on E, $(x, y) \mapsto f(x)g(y)$ defines a function on $E \times E$, which is denoted by $f \otimes g$.

We now bring in weak duality. However some of results are true even without duality and we shall not bother to indicate them explicitly. Assume that, with respect to a σ-finite measure m on E, X has a weak duality $\hat{X} = (\hat{X}_t, \hat{\mathsf{P}}^x)$, which is also a right process on $(E, \mathscr{B})$, with semigroup $(\hat{P}_t)$. As a convention the hat sign "ˆ" is always used on notations to indicate that they are with respect to $\hat{X}$. Clearly for $h \in \mathscr{S}$, $\hat{h} \in \hat{\mathscr{S}}$, X^h and $\hat{X}^{\hat{h}}$ are in duality with respect to the measure $h\hat{h}m$.

Let A be an additive functional of X, namely A is an increasing adapted process and for almost every $\omega \in \Omega$, (i) $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \geq 0$; (ii) $A_t(\omega) < \infty$ for $t < \zeta(\omega)$. Let U_A (resp. U_A^h) denote the potential operator of A under X (resp. X^h). Let also, for $\xi \in \text{Exc}$, ν_A^ξ the bivariate Revuz measure of A relative to X and ξ, precisely for any nonnegative measurable function G on $E \times E$,

$$\nu_A^{\xi}(G) := \uparrow \lim_{t \downarrow 0} \frac{1}{t} \mathsf{P}^{\xi} \int_0^t G(X_{s-}, X_s) dA_s.$$

Similarly the bivariate Revuz measure of A relative to X^h and an excessive measure ξ of X^h is denoted by $\nu_A^{\xi/h}$. Obviously the Revuz measure $\rho^x i_A$ of A relative to X and ξ is the right marginal measure of corresponding bivariate Revuz measure, i.e., $\rho_A^{\xi} = \nu_A^{\xi}(1 \otimes \cdot)$. We say A is integrable to ξ if ρ_A^{ξ} is finite, and s-integrable to xi if $A = \sum_n A^{(n)}$ where each $A^{(n)}$ is an additive functional of X integrable to ξ.

Lemma 4.4.1 If $h \in \mathscr{S}$ $\hat{h} \in \hat{\mathscr{S}}$ and A is s-integrable to m, then

$$\nu_A^{h\hat{h}m/h} = (\hat{h} \otimes h) \cdot \nu_A^m.$$

Proof. Let G be any non-negative measurable function on $E \times E$. Set

$$\kappa(]s,t]) := \int_{]s,t]} G(X_{u-}, X_u) dA_u,$$

for $0 \leq s < t$. Then κ is an s-integrable homogeneous random measure. By (4.8) in [20], we have

$$\nu_A^{h\hat{h}m/h}(G) = \rho_\kappa^{h\hat{h}m/h}(1_E) = \rho_\kappa^{\hat{h}m}(h) = \nu_A^{\hat{h}m}(1 \otimes h \cdot G).$$

Similarly by (8.12) in [23],

$$\begin{aligned} \nu_A^{\hat{h}m}(G) &= \lim_{t \downarrow 0} t^{-1} \mathsf{P}^m[\hat{h}(X_0)\kappa(]0,t])] \\ &= \lim_{t \downarrow 0} t^{-1} \mathsf{P}^m[\int_0^t \hat{h}(X_{s-})\kappa(ds)] \\ &= \nu_A^m(\hat{h} \otimes 1 \cdot G). \end{aligned}$$

Therefore

$$\begin{aligned} \nu_A^{h\hat{h}m/h} &= 1 \otimes h \cdot \nu_A^{\hat{h}m} \\ &= (1 \otimes h)(\hat{h} \otimes 1) \cdot \nu_A^m = \hat{h} \otimes h \cdot \nu_A^m. \end{aligned}$$

That completes the proof. □

Finally we take a look at the canonical measures of X and its h-transform. Let d be a metric on E and for positive integer n, $D_n := \{(x,y) \in E\times : d(x,y) \le \frac{1}{n}\}$,

$$A_t^{(n)} := \sum_{s\le t} 1_{D_n^c}(X_{s-}, X_s).$$

It is clear that each $A^{(n)}$ is an additive functional of X and s-integrable since the jumps are uniformly bounded. The canonical measure of X relative to m is the increasing limit of the bivariate Revuz measure of $A^{(n)}$ relative to X and m as n goes to infinity. Thus the following result is immediate from Lemma 4.4.1.

Corollary 4.4.1 Let ν^m and $\nu^{h\hat{h}m/h}$ be the canonical measures of X and X^h relative to m and $h\hat{h}m$, respectively. Then

$$\nu^{h\hat{h}m/h} = (\hat{h} \otimes h) \cdot \nu^m. \tag{4.4.2}$$

Remark. If (N, H) is a Lévy system of X, then it is not hard to check that a Lévy system of X^h can be taken as (N^h, H), where $N^h(x, dy) := N(x, dy)h(y)/h(x)$. The corollary above follows easily from this fact.

Let now X be an m-symmetric right process associated with a Dirichlet form $(\mathscr{E}, \mathscr{F})$. As usual $\mathscr{E}_q := \mathscr{E} + q(\cdot,\cdot)$ and $(\mathscr{E}_q, \mathscr{F})$ is nothing but the Dirichlet space associated with q-subprocess of X. It is known that the h-transform X^h is an h^2m-symmetric right process associated with a Dirichlet form $(\mathscr{E}^h, \mathscr{F}^h)$ which is defined to be

$$\begin{aligned} \mathscr{F}^h &:= \{u \in L^2(E; h^2m) : \ uh \in \mathscr{F}\}; \\ \mathscr{E}^h(u,v) &:= \mathscr{E}(uh, vh), \ u, v \in \mathscr{F}^h. \end{aligned} \tag{4.4.3}$$

The form $(\mathscr{E}^h, \mathscr{F}^h)$ is called the h-transform of $(\mathscr{E}, \mathscr{F})$. Clearly they are both quasi-regular and if h is bounded, then $b\mathscr{F} \subset \mathscr{F}^h$. For convenience any element in Dirichlet spaces assumes automatically its quasi-continuous version. Let $\mathscr{E}^{(c)}$, ν and k be the diffusion part of $\mathscr{E}$, the canonical measure and the killing measure of X relative to m. Then the Beurling-Deny formula for $\mathscr{E}$ reads as, for $u \in \mathscr{F}$,

$$\mathscr{E}(u,u) = \mathscr{E}^{(c)}(u,u) + \frac{1}{2}\int (u(x)-u(y))^2\nu(dx,dy) + k(u^2). \tag{4.4.4}$$

Let $\mathscr{E}^{h,(c)}$, ν^h and k^h be the counterparts of $(\mathscr{E}^h, \mathscr{F}^h)$.

Theorem 4.4.1 If $h \in \mathscr{S} \cap \mathscr{F}$ and $0 < h < \infty$ a.e. m, then for $u \in b\mathscr{F}^h$

$$\begin{aligned} \mathscr{E}^{h,(c)}(u,u) &= \mathscr{E}^{(c)}(uh,uh) - \mathscr{E}^{(c)}(u^2h,h); \\ \nu^h &= (h\otimes h)\cdot \nu; \\ k^h(u^2) &= \mathscr{E}(u^2h,h). \end{aligned}$$

Proof. . The second formula is immediate by Corollary 4.4.1. For the other two, since $h \in \mathscr{F}$, we see $1 \in \mathscr{F}^h$ and

$$\begin{aligned} k^h(u^2) &= \mathscr{E}^h(u^2,1) = \mathscr{E}(u^2h,h) \\ &= \mathscr{E}^{(c)}(u^2h,h) + \frac{1}{2}\int [u^2(x)h(x) - u^2(y)h(y)][h(x)-h(y)]\nu(dx,dy) \\ &+ k(u^2h^2) \\ &= \mathscr{E}^{(c)}(u^2h,h) + \int u^2(x)h(x)[h(x)-h(y)]\nu(dx,dy) + k(u^2h^2). \end{aligned}$$

Hence we have

$$\begin{aligned} \mathscr{E}^h(u,u) &= \mathscr{E}(uh,uh) \\ &= \mathscr{E}^{(c)}(uh,uh) + \frac{1}{2}\int [u(x)h(x) - u(y)h(y)]^2\nu(dx,dy) + k(u^2h^2) \\ &= \mathscr{E}^{(c)}(uh,uh) + \frac{1}{2}\int [u(x)-u(y)]^2 h(x)h(y)\nu(dx,dy) \\ &+ \frac{1}{2}\int [u^2(x)h(x) - u^2(y)h(y)][h(x)-h(y)]\nu(dx,dy) + k(u^2h\cdot h) \\ &= \mathscr{E}^{(c)}(uh,uh) - \mathscr{E}^{(c)}(u^2h,h) + \frac{1}{2}\int [u(x)-u(y)]^2\nu^h(dx,dy) + \mathscr{E}(u^2h,h). \end{aligned}$$

It follows immmediately that $\mathscr{E}^{h,(c)}(u,u) = \mathscr{E}^{(c)}(uh,uh) - \mathscr{E}^{(c)}(u^2h,h)$, of which the strong locality can be easily verified. □

Now we fix $\alpha > 0$ and consider the Ito-Watanabe's factorization of h-transforms. Let $h := U^\alpha g$ with $g \in L^2(E,m) \cap b\mathscr{B}(E)$ strictly positive and

(4.4.5) $$M_t := e^{-\alpha t}\frac{h(X_t)}{h(X_0)}.$$

Then M is a supermartingale multiplicative functional of X. The transformation carried by M is actually an h-transform for α-subprocess X^α of X. Let $M^{[h]}$ be the martingale part in Fukushima's decomposition of $A^{[h]} := h(X_\cdot) - h(X_0)$ and define formally (see §6.3 in [14] for details)

(4.4.6) $$Z_t^{[h]} := \int_0^t \frac{dM_s^{[h]}}{h(X_{s-})},$$

which is also a martingale additive functional of X. Let $Z^{[h]} = Z^{[h],c} + Z^{[h],d}$ be the decomposition as continuous and purely discontinuous parts. Denote by $L^{[h]}$ the Doleans-Dade's exponential martingale of $Z^{[h]}$. Then it admits a representation as follows

(4.4.7) $$L_t^{[h]} = \exp(Z_t^{[h],c} - \frac{1}{2}\langle Z^{[h],c}\rangle_t)e^{Z_t^{[h],d}}\prod_{s\le t}\frac{h(X_s)}{h(X_{s-})}e^{-(\frac{h(X_s)}{h(X_{s-})}-1)}1_{\{t<\zeta\}}.$$

By Ito's formula, we have

(4.4.8) $$\frac{h(X_t)}{h(X_0)} = L_t^{[h]}\cdot e^{\int_0^t \frac{Ah(X_s)}{h(X_s)}ds},$$

where A is the generator of $(\mathscr{E},\mathscr{F})$. Clearly $Ah = -g + \alpha h$. Let $B_t := \int_0^t \frac{g(X_s)}{h(X_s)}ds$. Then B is a PCAF and the Ito-Watanabe's factorization of M is

(4.4.9) $$M_t = L_t^{[h]}\cdot e^{-B_t}.$$

This factorization may be extended to $h \in \mathscr{S}^\alpha \cap \mathscr{F}$ (set of α-potentials in terms of [14]). In this case there exists a measure ξ of finite energy with a corresponding PCAF N such that $h = U^\alpha \xi = U_N^\alpha 1$ a.e. Then the factorization above holds with $B_t = \int_0^t \frac{1}{h(X_s)}dN_s$.

In some sense the transformation involving $L^{[h]}$ is more important than h-transform since $L^{[h]}$ may still be well defined as a martingale multiplicative functional and represented as (4.4.7) as long as h admits the Fukushima's decomposition, for instance when h is only a nonnegative element in $\mathscr{F}$ (though M, defined by (4.4.5), is no longer a supermaringale). By Lemma 6.3.1 of [14] the transformed process $\tilde{X}$ of X by $L^{[h]}$ is h^2m-symmetric. We denote by $(\tilde{\mathscr{E}}, \tilde{\mathscr{F}})$ the Dirichlet space on $L^2(E, h^2m)$ associated with $\tilde{X}$ and let $X^{\alpha,h}$ be the transformed process of X by M. Then $X^{\alpha,h}$ may be recovered from $\tilde{X}$ by a killing transform associated with B. The following result generalizes slightly Theorem 6.3.1 in [14].

Theorem 4.4.2 Let $h \in \mathscr{S}^\alpha \cap \mathscr{F}$ and be strictly positive.

(i) $\mathscr{F}^h$ is densely contained in $\tilde{\mathscr{F}}$.

(ii) for any $u \in b\mathscr{F}^h$,

$$\tilde{\mathscr{E}}(u,u) = \mathscr{E}^{h,(c)}(u,u) + \frac{1}{2}\int (u(x) - u(y))^2 h(x)h(y)\nu(dx,dy).$$

(iii) $1 \in \tilde{\mathscr{F}}$ and $\tilde{\mathscr{E}}(1,1) = 0$.

(iv) If, in addition, h is bounded, then $\mathscr{F} \subset \tilde{\mathscr{F}}$ and $u \in \mathscr{F}$,

$$\tilde{\mathscr{E}}(u,u) = \int h^2 d\mu^c_{\langle u\rangle} + \frac{1}{2}\int (u(x) - u(y))^2 h(x)h(y)\nu(dx,dy), \tag{4.4.10}$$

where $\mu^c_{\langle u\rangle}$ is the Revuz measure of $\langle M^{[u],c}\rangle$.

Proof. . It is easily seen that the Revuz measure of B relative to h^2m is $h\xi$ and the Dirichlet space associated with $X^{\alpha,h}$ is $(\mathscr{E}^h_\alpha, \mathscr{F}^h)$. By results in §6.1 of [14] we find that (i) is true and for any $u \in \mathscr{F}^h$,

$$\mathscr{E}^h_\alpha(u,u) = \tilde{\mathscr{E}}(u,u) + \xi(u^2h). \tag{4.4.11}$$

However by Theorem 4.4.1

$$\begin{aligned} \mathscr{E}^h_\alpha(u,u) &= \mathscr{E}^{h,(c)}(u,u) \\ &\quad + \frac{1}{2}\int (u(x) - u(y))^2 \nu^h(dx,dy) + \mathscr{E}(u^2h, h) + \alpha(hu, hu) \end{aligned} \tag{4.4.12}$$

and

$$\mathscr{E}(u^2h,h)+\alpha(hu,hu)=\mathscr{E}_\alpha(u^2h,h)=\mathscr{E}_\alpha(u^2h,U^\alpha\xi)=\xi(u^2h).$$

Then (ii) easily follows from (4.4.11) and (4.4.12). Finally $1\in\mathscr{F}^h\subset\tilde{\mathscr{F}}$ and $\tilde{\mathscr{E}}(1,1)=0$ obviously.

If h is bounded, it is easily seen that $b\mathscr{F}\subset\tilde{\mathscr{F}}$ and by Lemma 3.2.5 in [14] we find that for $u\in b\mathscr{F}$,

$$\mathscr{E}^{h,(c)}=\mathscr{E}^{(c)}(uh,uh)-\mathscr{E}^{(c)}(u^2h,h)=\int h^2d\mu^c_{\langle u\rangle}.$$

Therefore

$$\tilde{\mathscr{E}}(u,u)=\int h^2d\mu^c_{\langle u\rangle}+\frac{1}{2}\int(u(x)-u(y))^2h(x)h(y)\nu(dx,dy).$$

Then we know that $\tilde{\mathscr{E}}(u,u)\le||h||^2_\infty\mathscr{E}(u,u)$ for $u\in b\mathscr{F}$ and it follows that $\mathscr{F}\subset\tilde{\mathscr{F}}$ and (4.4.10) holds for $u\in\mathscr{F}$. That completes the proof. □

Remark. The problem was solved in §6.3 of [14], where $h=U^\alpha g$ with $g\in L^2(E,m)$ strictly positive and bounded. However our approach is rather different and more direct.

Example 1. Suppose that X is a diffusion, h is locally in $\mathscr{F}$ and strictly positive. Let $D_l(\mathscr{E})$ be the totality of functions locally in $\mathscr{F}$. Since the Fukushima's decomposition of $A^{[h]}$ still exists uniquely, we may still construct $L^{[h]}$ as above and

$$L^{[h]}_t=\exp\left(\int_0^t\frac{dM^{[h]}_s}{h(X_s)}-\frac{1}{2}\int_0^t\frac{d\langle M^{[h]}\rangle_s}{h^2(X_s)}\right)1_{\{t<\zeta\}}.$$

It is known by the recent work of Fitzsimmons [11] that $D_l(\mathscr{E})=D_l(\tilde{\mathscr{E}})$ and for $u\in D_l(\tilde{\mathscr{E}})$

$$\tilde{\mathscr{E}}(u,u)=\int h^2\mu_{\langle u\rangle}.$$

Example 2. More precisely let X be a Brownian motion on $\mathbf{R}^d$ and h a non-negative function locally in $H^1(\mathbf{R}^d)$. Set $l(x)=\ln h(x)$. Then l is also locally in $H^1(\mathbf{R}^d)$. It follows from Ito's formula that

$$L_t^{[h]} = \exp\left(\int_0^t \nabla l(X_s)\cdot dX_s - \frac{1}{2}\int_0^t |\nabla l(X_s)|^2 ds\right),$$

which gives us a drift (or distorted) Brownian motion Y on $\mathbf{R}^d$ which is h^2m-symmetric has the Dirichlet form $(\tilde{\mathscr{E}}, \tilde{\mathscr{F}})$ as for $u, v \in \tilde{\mathscr{F}}$,

$$\tilde{\mathscr{E}}(u,v) = \frac{1}{2}\int \nabla u\cdot\nabla v\cdot h^2 dx.$$

Moreover the generator of Y, when restricted to smooth functions of compact support, has the form

$$\tilde{A}f = \frac{1}{2}\Delta f + \nabla l\cdot\nabla f,$$

which is a drift to A. In particular when $l(x) = -\frac{1}{4}|x|^2$, Y is called the Ornstein-Ulenbeck's process on $\mathbf{R}^d$.

The examples hint that it is appropriate to call the transformation induced by $L^{[h]}$ the drift transformation and the corresponding Dirichlet form $(\tilde{\mathscr{E}}, \tilde{\mathscr{F}})$ the distorted form of $(\mathscr{E}, \mathscr{F})$ whenever it makes sense.

Chapter 5

Killing and Subordination

In the last chapter we formulated a so-called generalized Feynman-Kac formula for Dirichlet forms as follows. Let X be a Borel right process which is associated with a quasi-regular (non-symmetric) Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2 := L^2(E; m)$. Let M be a decreasing multiplicative functional of X and ν_M the bivariate Revuz measure of M relative to m. If (X, M) denotes the subprocess of X killed by M, then the Dirichlet form $(\mathscr{E}', \mathscr{F}')$ associated with (X, M) is given by

$$\begin{aligned} \mathscr{F}' &= \mathscr{F} \cap L^2(\rho_M) \cap L^2(\lambda_M); \\ \mathscr{F}'(u, v) &= \mathscr{E}(u, v) + \nu_M(u \otimes v), \ u, v \in \mathscr{F}', \end{aligned} \tag{5.0.1}$$

where ρ_M and λ_M are the right and left marginal measures of ν_M, respectively, and $u \otimes v(x, y) := u(x)v(y)$.

In this chapter we will study the inverse problem: given two quasi-regular Dirichlet forms $(\mathscr{E}, \mathscr{F})$ and $(\mathscr{E}', \mathscr{F}')$, what conditions make the process associated with $(\mathscr{E}', \mathscr{F}')$ be the subprocess of that with $(\mathscr{E}, \mathscr{F})$ killed by a decreasing multiplicative functional?

After collecting some standard definitions, notations and results in §5.1, we show in §5.2 that the subordination of Dirichlet forms is equivalent to the killing transform for Markov processes. In §5.3 we give a necessary and sufficient condition that a bivariate measure is a bivariate Revuz measure of a decreasing multiplicative functional.

5.1 Introduction

Throughout this paper we assume E to be a separable and metrizable topological space, $\mathscr{E}$ its Borel σ-algebra, m a σ-finite positive measure on $(E, \mathscr{E})$. Write $L^2 := L^2(E; m)$ and denote by $C_c(E)$ the set of all continuous functions on E with compact support. Let $(\mathscr{E}, \mathscr{F})$ be a Dirichlet form on L^2 defined as in [MR]. Particularly $\mathscr{F}$ is a dense linear subspace of $L^2(m)$ and the $\mathscr{E}_1$-norm on $\mathscr{F}$, defined by $\|u\|_{\mathscr{E}} := [\mathscr{E}_1(u,u)]^{\frac{1}{2}}$, satisfies the Markovian property: for any normal contraction ϕ it holds that $\phi(u) \in \mathscr{F}$ and $\|\phi(u)\|_{\mathscr{E}} \leq \|u\|_{\mathscr{E}}$. Set $\mathscr{F}^+ := \{u \in \mathscr{F};\ u \geq 0\}$. Define for $F \subset E$, F closed,

$$\mathscr{F}|_F := \{u \in \mathscr{F} :\ u = 0 \text{ a.e. } m \text{ on } F^c\}.$$

An increasing sequence $\{F_n\}$ of closed subsets of E is called an $\mathscr{E}$-nest if $\bigcup_n \mathscr{F}|_{F_n}$ is dense in $\mathscr{F}$ with respect to $\|\cdot\|_{\mathscr{E}}$. A subset $N \subset E$ is called $\mathscr{E}$-exceptional if $N \subset \bigcap_n F_n^c$ for some $\mathscr{E}$-nest $\{F_n\}$. We say that a property of points in E holds $\mathscr{E}$-quasi-everywhere ($\mathscr{E}$-q.e. in abbreviation) if the property holds off some $\mathscr{E}$-exceptional set. Given an $\mathscr{E}$-nest $\{F_n\}$ we define

$$C(\{F_n\}) := \{f :\ A \longrightarrow \mathbf{R},\ \bigcup_n F_n \subset A \subset E, f|_{F_n} \text{ is continuous for any } n \in \mathbf{N}\}.$$

An $\mathscr{E}$-q.e. defined function f on E is called $\mathscr{E}$-quasi-continuous if there exists an $\mathscr{E}$-nest $\{F_n\}$ such that $f \in C(\{F_n\})$.

Definition 5.1.1 Dirichlet form $(\mathscr{E}, \mathscr{F})$ is called quasi-regular if (i) there exists an $\mathscr{E}$-nest $\{E_n\}$ consisting of compact sets; (ii) there exists an $\mathscr{E}_1$-dense subset of $\mathscr{F}$ whose elements have $\mathscr{E}$-quasi-continuous m-versions; (iii) there exist $u_n \in \mathscr{F}$, $n \in \mathbf{N}$, having $\mathscr{E}$-quasi-continuous m-versions $\tilde{u}_n$, $n \in \mathbf{N}$, and an $\mathscr{E}$-exceptional set $N \subset E$ such that $\{\tilde{u}_n | n \in \mathbf{N}\}$ separates the points of $E \setminus N$.

An important result in [30] is that a Dirichlet form $(\mathscr{E}, \mathscr{F})$ is quasi-regular if and only if it is associated with a unique Borel right Markov process

$$X = (\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, X_t, \mathsf{P}^x)$$

on $(E, \mathscr{E})$, with sub-Markovian semigroup (P_t) and resolvent (U^q), as follows: for any $f \in L^2$, $u \in \mathscr{F}$ and $q > 0$, $U^q f$ is an $\mathscr{E}$-quasi-continuous element in $\mathscr{F}$ and $\mathscr{E}_q(u, U^q f) = (u, f)$. A set $B \in \mathscr{E}$ is called m-polar if $P^m(T_B < +\infty) = 0$, where T_B is the hitting time. It is easy to check that B is m-polar if and only if it is $\mathscr{E}$-exceptional. Moreover it is known that any exceptional set N is contained in a properly exceptional N_1 in the sense that $E \setminus N_1$ is absorbing. (See [Fu].) Having $(\mathscr{E}, \mathscr{F})$ at hands, we define a notion of multiplicative functionals which is a little weaker than what is usually used in theory of Markov processes.

Definition 5.1.2 A real valued function $M = (M_t(\omega) : t \geq 0, \omega \in \Omega)$ is called a decreasing multiplicative functional of $(\mathscr{E}, \mathscr{F})$ (or X) (MF in abbreviation) if (i) $M_t(\cdot)$ is $(\mathscr{F}_t)$-adapted; (ii) there exists a set $\Lambda \in \mathscr{F}_\infty$ and a properly $\mathscr{E}$-exceptional set $N \subset E$ such that $\mathsf{P}^x(\Lambda) = 1$ for all $x \in E \setminus N$, $\theta_t \Lambda \subset \Lambda$ for all $t > 0$ and moreover for each $\omega \in \Lambda$, $M_\cdot(\omega)$ is right continuous on $[0, \infty]$, $0 \leq M_t(\omega) \leq 1$ for any $t < \zeta(\omega)$, $M_0(\omega) = 1$ and $M_{t+s}(\omega) = M_s(\omega) M_t(\theta_s \omega)$ for any $t, s \geq 0$.

Clearly M is a perfect MF in the ordinary sense but with respect to the restricted Borel right process $X|_{E-N'}$ for a properly m-polar Borel set N'. This makes no differences in the view of P^m. Hence we can use almost all results on bivariate Revuz measures and related analysis developed in Chapter 4 freely. Given $M \in$MF the bivariate Revuz measure of M relative to m is defined as

$$(5.1.1) \qquad \nu_M(F) := \uparrow \lim_{t \downarrow 0} \frac{1}{t} \mathsf{P}^m [\int_0^t F(X_{s-}, X_s) d(-M_s)], \; F \in (\mathscr{E} \times \mathscr{E})^+,$$

and is actually that of $(1 - M_t)$, which is an M-additive functional or an additive functional of M in some books. (Refer to [13] for existence of Revuz measures in this case.) Denote by ρ_M and λ_M the right and left marginal measures of ν_M, respectively. We also get a right process with transition semigroup (Q_t) given by $Q_t f := \mathsf{P}^\cdot [f(X_t) M_t]$ for any $f \in \mathscr{E}$, and the corresponding resolvent is denoted by (V^q). This process is usually called the subprocess of X killed by M and denoted by (X, M). The condition that $M_0 \equiv 1$ amounts to the assumption that the state space of the killed process is the same as that of the original process. Ruled out here is killing at the first hitting time of a non-m-polar Borel set. We know from the work of Silverstein [40] that in the symmetric case, if $(\mathscr{E}', \mathscr{F}')$ is obtained by killing the process associated with $(\mathscr{E}, \mathscr{F})$ at the first hitting time of a Borel set, then $\mathscr{F}' \subset \mathscr{F}$, $\mathscr{E}' = \mathscr{E}$ on $\mathscr{F}' \times \mathscr{F}'$, and $\mathscr{F}' \cap L^\infty$ is an ideal in the algebra $\mathscr{F} \cap L^\infty$. Moreover the converse assertion is also true. This result of Silverstein can certainly be extended to near-symmetry context of this paper, therefore the main results following can be extended to accommodated the most general killing transformation without great difficulties. But to keep the article in a reasonable length, I won't give details here.

If X is associated with a quasi-regular Dirichlet form $(\mathscr{E}, \mathscr{F})$, it has automatically a weak dual $\hat{X}$ relative to m which is associated with the dual form $(\hat{\mathscr{E}}, \hat{\mathscr{F}})$ of $(\mathscr{E}, \mathscr{F})$ defined by $\mathscr{F}' := \mathscr{F}$ and $\hat{\mathscr{E}}(u, v) := \mathscr{E}(v, u),\ u, v \in \hat{\mathscr{F}}$. As a convention the notations with 'hat' refer to $\hat{X}$ and assume the same meanings as to X. It is known from §3.6 that ν_M is dual to $\hat{\nu}_{\hat{M}}$; i.e.,

$$\hat{\nu}_{\hat{M}}(dx, dy) = \nu_M(dy, dx), \tag{5.1.2}$$

and the generalized Revuz formula holds

$$\nu_M(\hat{V}^1 f \otimes g) = (f, U_M^1 g),\ \ f, g \in \mathscr{E}^+, \tag{5.1.3}$$

where $(\hat{V}^q)$ is the resolvent of $(\hat{X}, \hat{M})$ and U_M^q the potential operator of $(1 - M_t)$.

5.2 Subordination and strong subordination

We begin this section with a definition.

Definition 5.2.1 Given two Dirichlet forms $(\mathscr{E}, \mathscr{F})$ and $(\mathscr{E}', \mathscr{F}')$ on L^2, we say that $(\mathscr{E}', \mathscr{F}')$ is subordinate to $(\mathscr{E}, \mathscr{F})$ if $\mathscr{F}' \subset \mathscr{F}$ and $b(u, v) \geq a(u, v)$ for any $u, v \in \mathscr{F}'^+$, and $(\mathscr{E}', \mathscr{F}')$ is strongly subordinate to $(\mathscr{E}, \mathscr{F})$ if, in addition, $\mathscr{F}'$ is dense in $\mathscr{F}$ with $\mathscr{E}_1$-norm.

The following lemma is essential in later discussions.

Lemma 5.2.1 If $(\mathscr{E}', \mathscr{F}')$ is subordinate to $(\mathscr{E}, \mathscr{F})$, then there exists an $C > 0$ such that $\|u\|_{\mathscr{E}} \leq C \cdot \|u\|_{\mathscr{E}'}$ for any $u \in \mathscr{F}'$.

Proof. Let I be the inclusion operator from $(\mathscr{F}', \|\cdot\|_{\mathscr{E}'})$ to $(\mathscr{F}, \|\cdot\|_{\mathscr{E}})$; i.e., $I(u) = u$ for any $u \in \mathscr{F}'$. It suffices to show that I is continuous. Let $\{u_n\} \subset \mathscr{F}'$ and $u_n \longrightarrow 0$ in $\mathscr{E}'_1$-norm. By the Markovian property of $(\mathscr{E}', \mathscr{F}')$ we find that $u_n^+ \longrightarrow 0$ and $u_n^- \longrightarrow 0$ both in $\mathscr{E}'_1$-norm. Since $(\mathscr{E}', \mathscr{F}')$ is subordinate to $(\mathscr{E}, \mathscr{F})$, $\|u_n^+\|_{\mathscr{E}} \le \|u_n^+\|_{\mathscr{E}'}$ and $\|u_n^-\|_{\mathscr{E}} \le \|u_n^-\|_{\mathscr{E}'}$. Thus $u_n^+ \longrightarrow 0$ and $u_n^- \longrightarrow 0$ both in $\mathscr{E}_1$-norm, and then we have $u_n \longrightarrow 0$ in $\mathscr{E}_1$-norm. □

Corollary 5.2.1 If $(\mathscr{E}', \mathscr{F}')$ is strongly subordinate to $(\mathscr{E}, \mathscr{F})$, then any $\mathscr{E}'$-nest is an $\mathscr{E}$-nest. Therefore any $\mathscr{E}'$-quasi-continuous function is $\mathscr{E}$-quasi-continuous and that $(\mathscr{E}', \mathscr{F}')$ is quasi-regular implies that $(\mathscr{E}, \mathscr{F})$ is too.

Proof. Let $\{F_n\}$ be a $\mathscr{E}'$-nest. Then $\mathscr{D} := \bigcup_n \mathscr{F}'|_{F_n}$ is dense in $(\mathscr{F}', \|\cdot\|_{\mathscr{E}'})$. By Lemma 5.2.1 we know that $\mathscr{D}$ is dense in $(\mathscr{F}', \|\cdot\|_{\mathscr{E}})$. Also $\mathscr{F}'$ is dense in $(\mathscr{F}, \|\cdot\|_{\mathscr{E}})$. Hence $\mathscr{D}$ is dense in $(\mathscr{F}, \|\cdot\|_{\mathscr{E}})$ and so is $\bigcup_n \mathscr{F}|_{F_n}$, since it contains $\mathscr{D}$; i.e., $\{F_n\}$ is an $\mathscr{E}$-nest. □

We shall first prove that killing transform implies subordination.

Theorem 5.2.1 Let X be a Borel right process associated with a quasi-regular Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2(m)$. If $M \in$ MF, then (i) the bilinear form $(\mathscr{E}', \mathscr{F}')$ defined in (5.0.1) is a quasi-regular Dirichlet form on $L^2(m)$ with which the subprocess (X, M) is associated; (ii) $\mathscr{F}'$ is dense in $\mathscr{F}$ with $\mathscr{E}_1$-norm. Therefore $(\mathscr{E}', \mathscr{F}')$ is strongly subordinate to $(\mathscr{E}, \mathscr{F})$.

Proof. It was shown in Chapter 4 that $(\mathscr{E}', \mathscr{F}')$ is a Dirichlet form on $L^2(m)$ with which (X, M) is associated. Hence it suffices to show that (i) $(\mathscr{E}', \mathscr{F}')$ is quasi-regular and (ii) $\mathscr{F}'$ is dense in $(\mathscr{F}, \|\cdot\|_{\mathscr{E}})$. But the proof of (i) is a step by step exercise if we follow that of IV.4.5 and IV 4.6 of [30] and replace $e^{-A_t^\mu}$ there by M_t. We will not write them here explicitly.

(ii) Fix $f \in \mathscr{E}$ with $0 < f \le 1$ and $m(f) < \infty$. By (5.1.2) and (5.1.3) we have

$$\lambda_M(\hat{V}^1 f) = \nu_M(V^1 f \otimes 1) = (f, U_M^1 1) \le m(f) < \infty;$$
$$\rho_M(V^1 f) = \nu_M(1 \otimes V^1 f) = \hat{\nu}_{\hat{M}}(V^1 f \otimes 1) = (f, \hat{U}_{\hat{M}} 1) \le m(f) \le \infty,$$

where $(\hat{V}^q)$ and (V^q) are resolvents of $(\hat{X}, \hat{M})$ and (X, M), respectively. Set $F_n := \{x : V^1 f(x) \ge \frac{1}{n}, \hat{V}^1 f(x) \ge \frac{1}{n}\}$. Then F_n is finely closed, $E - \bigcup_n F_n$

is an $\mathscr{E}$-exceptional set, and both $\rho_M(F_n)$ and $\lambda_M(F_n)$ are finite for each $n \in \mathbf{N}$.

Now let $g \in L^2(m)$ and be nonnegative, bounded, and $u := U^1 g$, $u_n := u - P^1_{F_n^c} u$, where $P^1_{F_n^c}$ is the balayage operator. It is clear that

$$u_n = \mathsf{P}^{\cdot} \int_0^{T_n} e^{-t} g(X_t) dt,$$

where $T_n := T_{F_n^c}$, and hence $u_n = 0$ $\mathscr{E}$-q.e. on F_n^c. Obviously we have $u_n \in \mathscr{F} \cap L^2(\rho_M) \cap L^2(\lambda_M)$ or $u_n \in \mathscr{F}'$ for each n. Since $T_n \uparrow \zeta$ a.e. P^m, $u_n \longrightarrow u$ a.e.-m. On the other hand $\mathscr{E}_1(u_n, u_n) = \mathscr{E}_1(u - P^1_{F_n^c}, u) \leq \mathscr{E}_1(u, u)$. Hence by I.2.12 of [30] there exists a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}$ such that its Cesaro mean $\frac{1}{n}\sum_{k=1}^n u_{n_k} \longrightarrow u$ in $\mathscr{E}_1$-norm. This gives the conclusion that $\mathscr{F}'$ is dense in $U^1(L^2)$ with respect to $\mathscr{E}_1$-norm, but the later is dense in $(\mathscr{F}, \|\cdot\|_{\mathscr{E}})$. Therefore $\mathscr{F}'$ is dense in $(\mathscr{F}, \|\cdot\|_{\mathscr{E}})$. □

Remark. The idea of Theorem 5.2.1(ii) comes from an unpublished note of Fitzsimmons, who proved this denseness in the case that $(\mathscr{E}, \mathscr{F})$ is symmetric and M is continuous and never vanishes. But it is this denseness that plays a very important role in exploring the inverse problem.

The converse to Theorem 5.2.1 is also true and can be stated as follows.

Theorem 5.2.2 Let X and Y be two Borel right processes which are associated with two quasi-regular Dirichlet forms $(\mathscr{E}, \mathscr{F})$ and $(\mathscr{E}', \mathscr{F}')$, respectively. If $(\mathscr{E}', \mathscr{F}')$ is strongly subordinate to $(\mathscr{E}, \mathscr{F})$, then there exists an $M \in$ MF such that Y is the subprocess of X killed by M.

Proof. Let $f, g \in L^2(m)$ be nonnegative Borel measurable functions, and let $(\hat{U}^q)$, (U^q) and (V^q) be the resolvents of $\hat{X}$, X and Y, respectively. Since $\hat{U}^q f \in \mathscr{F}$ for $q > 0$, we can choose $v_n \in \mathscr{F}'$, $n \in \mathbf{N}$, with $v_n \longrightarrow \hat{U}^q f$ in $\mathscr{E}_1$-norm. Then $v_n^+ \longrightarrow \hat{U}^q f$ in $\mathscr{E}_1$-norm and clearly, in $L^2(m)$ norm. Now $\mathscr{E}'_q(v_n^+, V^q g) \geq \mathscr{E}_q(v_n^+, V^q g)$ for any $n \in \mathbf{N}$. Hence $(v_n^+, g) \geq \mathscr{E}_q(v_n^+, V^q g)$ and then $(\hat{U}^q f, g) \geq \mathscr{E}_q(\hat{U}^q f, V^q g) = (f, V^q g)$ with n tending to infinity. By duality we have $(f, U^q g) \geq (f, V^q g)$ for any $q > 0$; in particular, $U^q g \geq V^q g$ m-a.e. for $q > 0$ and $g \in C_c(E)$, beinng non-negative. Let us take $\{E_k\}$ as a $\mathscr{E}'$-nest consisting of compact sets and $F := \bigcup_k E_k$. Then $\{E_k\}$ is an $\mathscr{E}$-nest.

By IV.3.2(iii) and the proof of IV.3.5 of [30] we can see that (i) there exists a metric ρ on Y, compatible with the original topology, such that each E_k is compact with ρ, and hence F and $C_c(F)$ are separable; (ii) both X and Y actually live on F. Let $\{q_i\}$ be a dense set in $(0,\infty)$ and $\{g_j\}$ a dense set in $C_c^+(F)$. Since each $V^{q_i}g_j$ is $\mathscr{E}'$-quasi-continuous, it is $\mathscr{E}$-quasi-continuous by Corollary 5.2.1. Then it follows that there exists an $\mathscr{E}$-nest $\{F_n\}$ such that $\{U^{q_i}g_j, V^{q_i}g_j : \ i,j \in \mathbf{N}\} \subset C(\{F_n\})$. Hence $U^{q_i}g_j(x) \geq V^{q_i}g_j(x)$ for any $x \in \bigcup F_n$ and $i,j \in \mathbf{N}$. Let $N := \cap F_n^c$, which is clearly an $\mathscr{E}$-exceptional set. By the continuity of $U^{\cdot}f(x)$ and $V^{\cdot}f(x)$ for $f \in C_c(F)$ and $x \in E$, we find that $U^q g_j(x) \geq V^q g_j(x)$ for any $q > 0$, $j \in \mathbf{N}$ and $x \in F - N$. The similar reasoning gives

$$U^q g(x) \geq V^q g(x), \quad q > 0, \ g \in C_c(Y), \ x \in F - N. \tag{5.2.1}$$

Thus (V^q) is exactly subordinate to (U^q) (in terminology of [3] and [37]) and by [37] there exists $M \in$MF such that

$$V^q f(x) = \mathsf{P}^x \int_0^\infty e^{-qt} M_t f(X_t) dt, \ x \in F - N, \ f \in b\mathscr{E}^+, q \geq 0. \tag{5.2.2}$$

It follows that Y is the subprocess of X killed by M. □

A Dirichlet form $(\mathscr{E}', \mathscr{F}')$ on $L^2(m)$ is called reflectly subordinate to $(\mathscr{E}, \mathscr{F})$ if $\mathscr{F}' \subset \mathscr{F}$ and $\mathscr{E}' = \mathscr{E}$ on $\mathscr{F}'$. A simple example of reflect subordination is a Brownian motion on a bounded domain $D \subset \mathbf{R}^n$ with its so-called reflect Brownian motion on boundary. The following corollary is a result on the subordination. It also tells that a general subordination can be decomposed into combination of a strong subordination and a reflect subordination.

Corollary 5.2.2 Let $(\mathscr{E}, \mathscr{F})$ and $(\mathscr{E}', \mathscr{F}')$ be two Dirichlet forms on L^2. If $(\mathscr{E}', \mathscr{F}')$ is quasi-regular and subordinate to $(\mathscr{E}, \mathscr{F})$, then there exists a measure σ on $E \times E$ such that

$$\mathscr{E}'(u,v) = \mathscr{E}(u,v) + \sigma(u \otimes v), \ u,v \in \mathscr{F}'.$$

Proof. Let $\bar{\mathscr{F}}'$ be the closure of $\mathscr{F}'$ in $(\mathscr{F}, \|\cdot\|_{\mathscr{E}})$. Define $\bar{\mathscr{E}}'(u,v) := \mathscr{E}(u,v)$ for any $u, v \in \bar{\mathscr{F}}'$. Clearly $(\bar{\mathscr{E}}', \bar{\mathscr{F}}')$ is a Dirichlet form on $L^2(m)$ and by Corollary 5.2.1 it is quasi-regular. It follows from Theorem 5.2.2 that there exists an $M \in$MF such that $\mathscr{E}'(u,v) = \bar{\mathscr{E}}'(u,v) + \nu_M(u \otimes v) = \mathscr{E}(u,v) + \sigma(u \otimes v)$ for $u, v \in \mathscr{F}'$, where σ is taken to be ν_M. □

Remark. Let $(A, D(A))$ and $(B, D(B))$ be the respective generators of $(\mathscr{E}, \mathscr{F})$ and $(\mathscr{E}', \mathscr{F}')$. Benyaiche [Be] proved that in the case of classical symmetry if $(\mathscr{E}', \mathscr{F}')$ is subordinate to $(\mathscr{E}, \mathscr{F})$ and $D(A) \cap D(B)$ is dense in $C_c(E)$, then the conclusion of Corollary 3.6 holds.

5.3 Characterization of bivariate smooth measures

From the transfer method developed in Chapter VI of [30] we know that a quasi-regular Dirichlet form $(\mathscr{E}, \mathscr{F})$ admits a Beurling-Deny type decomposition along the diagonal of $\mathscr{F} \times \mathscr{F}$,

(5.3.1)

$$\mathscr{E}(u,u) = \mathscr{E}^c(u,u) + \frac{1}{2}\int_{E\times E}[u(x)-u(y)]^2 J(dx,dy) + \int_E (u(x))^2 k(dx),\ u \in \mathscr{F},$$

where $\mathscr{E}^c$ is the diffusion part, J the jumping measure and k the killing measure of $(\mathscr{E}, \mathscr{F})$. If X is the Borel right process associated with $(\mathscr{E}, \mathscr{F})$, then J coincides with the canonical jumping measure ν of X relative to m, which is defined by

$$(5.3.2) \qquad \nu(F) := \lim_{t\to 0} \frac{1}{t} \mathsf{P}^m \sum_{s\le t} F(X_{s-}, X_s) 1_{\{X_{s-}\neq X_s\}},\ F \in (\mathscr{E} \times \mathscr{E})^+.$$

A non-negative real process $A = (A_t)_{t\ge 0}$ on Ω is said to be a positive continuous additive functional (PCAF in abbreviation) of X if it satisfies IV(4.6) in [30], and a positive measure μ on $(E, \mathscr{E})$ is said to be smooth if μ does not charge $\mathscr{E}$-exceptional sets and there exists an $\mathscr{E}$-nest $\{F_n\}$ of compact sets such that $\mu(F_n) < \infty$ for each n. It is known that μ is smooth if and only if it is the Revuz measure of a PCAF (see Chapter IV of [30]).

Recently there were many works on characterization of smooth measures. As an application of Theorem 5.2.2 we will give a characterization which was proved in [45] by different but rather analytic approaches. Let $\tilde{u}$ stand for an $\mathscr{E}$-quasi-continuous m-version of $u \in \mathscr{F}$. Given a positive measure μ not charging m-polar sets we define

$$
\begin{aligned}
\mathscr{F}' &:= \mathscr{F} \cap L^2(\mu), \\
\mathscr{E}'(u, v) &:= \mathscr{E}(\tilde{u}, \tilde{v}) + \mu(\tilde{u} \cdot \tilde{v}), \; u, v \in \mathscr{F}'.
\end{aligned}
\tag{5.3.3}
$$

Then $(\mathscr{E}', \mathscr{F}')$ is a well-defined bilinear form on $L^2(m)$. It is actually a Dirichlet form in wide sense, namely, $\mathscr{F}'$ may not be dense in L^2.

Theorem 5.3.1 Let $(\mathscr{E}, \mathscr{F})$ be a quasi-regular Dirichlet form and μ a positive measure not charging m-polar sets. Then μ is smooth if and only if $(\mathscr{E}', \mathscr{F}')$ defined in (5.3.3) is a quasi-regular Dirichlet form on $L^2(m)$ and $\mathscr{F}'$ is dense in $(\mathscr{F}, \| \cdot \|_{\mathscr{E}})$.

Proof. The 'only if' is a direct consequence of Theorem 5.2.1. Conversely by Theorem 5.2.2 there exists $M \in$MF such that $\nu_M(dx, dy) = \delta_x(dy)\mu(dx)$ where δ_x is the singleton at x. By the representation theorem of ν_M in §3.4, $\nu_S = 0$ where $S(\omega) := \inf\{t > 0 : M_t > 0\}$. Also by the equivalence theorem in §3.6 we have $S = \zeta$ a.s. P^m; i.e. M never vanishes (before ζ). Similar arguments show that the pure jump factor of M is m-equivalent to 1. Hence M is continuous and never vanishes. Let $A_t := \log M_t$. Then $A = (A_t)$ is a PCAF and μ is nothing but the Revuz measure of A. □

The following lemma is easy to check.

Lemma 5.3.1 Let $(\mathscr{E}, \mathscr{F})$ and $(\mathscr{E}', \mathscr{F}')$ be two Dirichlet forms on $L^2(m)$. If $\mathscr{F} = \mathscr{F}'$ and $\| \cdot \|_{\mathscr{E}}$ is equivalent to $\| \cdot \|_{\mathscr{E}'}$, then $(\mathscr{E}, \mathscr{F})$ is quasi-regular if and only if $(\mathscr{E}', \mathscr{F}')$ is.

Given a positive measure σ on $E \times E$ let σ_r be its right marginal measure, σ_l its left marginal measure and $\bar{\sigma} := \frac{1}{2}(\sigma_r + \sigma_l)$. We call σ a bivariate smooth measure if (i) $\bar{\sigma}$ is smooth; (ii) $\sigma|_{D^c} \leq J$, where D is the diagonal of $E \times E$.

Theorem 5.3.2 Let X be a Borel right process associated with a quasi-regular Dirichlet form $(\mathscr{E}, \mathscr{F})$. A positive measure σ on $E \times E$ is a bivariate smooth measure if and only if σ is the bivariate Revuz measure of an $M \in$MF.

Proof. We will first show 'if' part. Assume that $\sigma = \nu_M$ for an $M \in$ MF.

It is easy to see that σ_r and σ_l do not charge m-polar sets and neither does $\bar{\sigma}$. The assertion that $\sigma|_{D^c} \leq J$ is clear from the definition. Let $(\mathscr{E}', \mathscr{F}')$ be the Dirichlet form associated with (X, M). Then by Theorem 5.2.1 $(\mathscr{E}', \mathscr{F}')$ is quasi-regular and strongly subordinate to $(\mathscr{E}, \mathscr{F})$. Define $\bar{\mathscr{F}}' := \mathscr{F}'$ and $\bar{\mathscr{E}}'(u, v) := \mathscr{E}(u, v) + \bar{\sigma}(u \otimes v)$. Clearly $\mathscr{E}'(u, u) \leq \bar{\mathscr{E}}'(u, u)$ for any $u \in \mathscr{F}'$. On the other hand let $\{u_n\} \subset \mathscr{F}'$ be a sequence which converges to 0 in $\| \cdot \|_{\mathscr{E}'}$. Since

$$\mathscr{E}'(u, u) = \mathscr{E}^c(u, u) + \int [u(x) - u(y)]^2 (J - \sigma)(dx, dy) + (\bar{\sigma} + k)(u^2)$$

for $u \in \mathscr{F}'$, $\lim_n \bar{\sigma}(u_n^2) = 0$ and $\lim_n \int [u_n(x) - u_n(y)]^2 \sigma(dx, dy) \leq \lim_n 4\bar{\sigma}(u_n^2) = 0$. Hence $\{u_n\}$ converges to 0 in $\mathscr{E}_1$-norm and in $\bar{\mathscr{E}}_1'$-norm; i.e., there exists a constant $M > 0$ such that $\bar{\mathscr{E}}'(u, u) \leq M \cdot \mathscr{E}'(u, u)$ for any $u \in \mathscr{F}'$. Then $\mathscr{E}_1'$-norm is equivalent to $\bar{\mathscr{E}}'$-norm. Thus $(\bar{\mathscr{E}}', \bar{\mathscr{F}}')$ is quasi-regular and strongly subordinate to $(\mathscr{E}, \mathscr{F})$. By Lemma 5.3.1 $\bar{\sigma}$ is smooth.

Conversely if σ is a bivariate smooth measure, we define $\mathscr{F}' := \mathscr{F} \cap L^2(\bar{\sigma})$, $\mathscr{E}'(u, v) := \mathscr{E}(u, v) + \sigma(u \otimes v)$ and as above. Then $(\bar{\mathscr{E}}', \bar{\mathscr{F}}')$ is quasi-regular and strongly subordinate to $(\mathscr{E}, \mathscr{F})$. Hence by the arguments above we can show that $(\mathscr{E}', \mathscr{F}')$ is also quasi-regular and strongly subordinate to $(\mathscr{E}, \mathscr{F})$. Finally Theorem 5.2.2 tells that σ is a bivariate Revuz measure of some $M \in$MF. □

5.4 Regular subspaces of Brownian motion

Let X be a locally compact separable metric space and m a fully supported positive Radon measure on X. $C_0(X)$ will denote the space of continuous functions on X with compact support. A Markovian symmetric closed form $(\mathscr{E}, \mathscr{F})$ on $L^2(X; m)$ is called a *Dirichlet form*, while it is called *regular* if $\mathscr{F} \cap C_0(X)$ is dense both in $C_0(X)$ with uniform norm and in $\mathscr{F}$ with $\mathscr{E}_1$-norm. The regularity guarantees the existence of a unique associated symmetric Hunt process on X.

Assume that $(\mathscr{E}, \mathscr{F})$ is a regular Dirichlet form on $L^2(X, m)$ and $\hat{\mathscr{F}}$ a linear subspace of $\mathscr{F}$. If $(\mathscr{E}, \hat{\mathscr{F}})$ is also a regular Dirichlet form on $L^2(X, m)$, it is natural to ask whether or not $\hat{\mathscr{F}}$ coincides with $\mathscr{F}$. If not, what condition would guarantee the coincidence? The question was originally raised in §5.2 where it was asked if strong subordination is equivalent to subordination.

Throughout this section let I be a finite open interval (a,b) or the real line $\mathbf{R}$. Denote by $L^2(I)$ the space of square integrable functions on I and we let

$$H^1(I) = \{u \in L^2(I) : u \text{ is absolutely continuous and } u' \in L^2(I)\}.$$

$$\mathbf{D}(u,v) = \int_I u' \cdot v' dx \qquad u, v \in H^1(I).$$

$(H^1(I), \frac{1}{2}\mathbf{D})$ can be considered as a regular local recurrent Dirichlet form on $L^2(\bar{I})$, where $\bar{I}$ denotes $[a,b]$ (resp. $\mathbf{R}$) for $I = (a,b)$ (resp. $I = \mathbf{R}$). The associated diffusion process on $\bar{I}$ is the reflecting Brownian motion (resp. the Brownian motion).

We call $(\mathscr{E}, \mathscr{F})$ a *Dirichlet subspace of* $(H^1(I), \frac{1}{2}\mathbf{D})$ if

$$\mathscr{F} \subset H^1(I), \qquad \mathscr{E}(u,v) = \frac{1}{2}\mathbf{D}(u,v),\ u, v \in \mathscr{F}, \tag{5.4.1}$$

and $(\mathscr{F}, \mathscr{E})$ is a Dirichlet form on $L^2(I)$. It is called *regular on* $L^2(\bar{I})(= L^2(I))$ if $\mathscr{F} \cap C(\bar{I})$ is dense both in $\mathscr{F}$ and $C(\bar{I})$, where $C(\bar{I})$ denotes the space of continuous functions on $\bar{I}$. It is called *recurrent* if its extended Dirichlet space $\mathscr{F}_e$ contains the constant function 1. When I is finite, any regular Dirichlet subspace of $(H^1(I), \frac{1}{2}\mathbf{D})$ is automatically recurrent.

In this paper, we shall prove that the Sobolev space $(H^1(I), \frac{1}{2}\mathbf{D})$ admits as its regular Dirichlet subspaces the following family of spaces $(\mathscr{F}^{(\mathtt{s})}, \mathscr{E}^{(\mathtt{s})})_{\mathtt{s} \in \mathbf{S}}$:

$$\mathscr{F}^{(\mathtt{s})} := \{u \in L^2(I; dx) : u \text{ is absolutely continuous with respect to } d\mathtt{s}(x), \int_I \left(\frac{du}{d\mathtt{s}}(x)\right)^2 d\mathtt{s}(x) < \infty\} \tag{5.4.2}$$

$$\mathscr{E}^{(\mathtt{s})}(u,v) := \frac{1}{2}\int_I \frac{du}{d\mathtt{s}}\frac{dv}{d\mathtt{s}} d\mathtt{s}, \quad u, v \in \mathscr{F}^{(\mathtt{s})}, \tag{5.4.3}$$

for $\mathtt{s}$ belonging to the space of functions

(5.4.4)

$$\mathbf{S} = \{\mathtt{s} : \mathtt{s}(x) \text{ is absolutely continuous, strictly increasing in } x \in I \text{ and } \mathtt{s}'(x) = 0 \text{ or } 1 \text{ for a.e. } x \in I,\ \mathtt{s}(\eta) = 0\},$$

where η denotes either a or 0 according as I is (a, b) or $\mathbf{R}$.

We shall further consider the subfamily

$$\hat{\mathbf{S}} = \begin{cases} \mathbf{S} & \text{when } \quad I = (a, b), \\ \{\mathtt{s} \in \mathbf{S} : \mathtt{s}(\pm\infty) = \pm\infty\} & \text{when } \quad I = \mathbf{R}, \end{cases} \tag{5.4.5}$$

of $\mathbf{S}$ and prove that all recurrent regular Dirichlet subspaces of $(H^1(I), \frac{1}{2}\mathbf{D})$ are exhausted by the family of spaces $(\mathscr{F}^{\mathtt{s}}, \mathscr{E}^{\mathtt{s}})_{\mathtt{s}\in\hat{\mathbf{S}}}$.

For $\mathtt{s} \in \mathbf{S}$, we let $E_{\mathtt{s}} = \{x \in I : \mathtt{s}'(x) = 0\}$ and denote by $|\cdot|$ the Lebesgue measure. Denote by φ the linear function $\varphi(x) = x$, $x \in I$. Clearly, $\varphi \in \mathscr{F}^{(\mathtt{s})}$ ($\mathscr{F}^{(\mathtt{s})}_{loc}$ when $I = \mathbf{R}$) if and only if $|E_{\mathtt{s}}| = 0$, or equivalently, the inverse function of $\mathtt{s}$ is absolutely continuous. In this case, $\mathtt{s}(x)$ equals either $\varphi(x) - a$ or $\varphi(x)$ according as I is (a, b) or $\mathbf{R}$, and $\mathscr{F}^{(\mathtt{s})} = H^1(I)$ of course. A typical example of an element $\mathtt{s} \in \mathbf{S}$ for $I = (0, 2)$ with $|E_{\mathtt{s}}| > 0$ is provided by

$$\mathtt{s} := \mathtt{t}^{-1}, \qquad \mathtt{t}(x) := c(x) + x,\ x \in (0, 1), \tag{5.4.6}$$

where c is the standard Cantor function on $(0, 1)$.

In this connection, we would like to mention that the second and the third authors have considered in [15] a slightly more general regular Dirichlet form than $(H^1(I), \frac{1}{2}\mathbf{D})$ for $I = (0, 1)$ and studied its regular Dirichlet subspace. Unfortunately, there is a flaw in the proof of Theorem 2 in it. We take this opportunity to correct Theorem 2 in [15] to the following weaker assertion for which the proof given in [15] works: *Let $\check{\mathscr{F}}$ be a subspace of $\mathscr{F}$ such that $(\mathscr{E}, \check{\mathscr{F}})$ is a regular Dirichlet space on $L^2(\bar{I}, \rho dx)$. Assume that a scale function $\mathtt{s}$ of the diffusion process on $\bar{I}$ associated with $(\mathscr{E}, \check{\mathscr{F}})$ admits an absolutely continuous inverse $\mathtt{t}$. Then $\check{\mathscr{F}} = \mathscr{F}$.*

In §3, we shall construct a recurrent diffusion process on $[a, b]$ (resp. $\mathbf{R}$) associated with the space $(\mathscr{F}^{\mathtt{s}}, \mathscr{E}^{\mathtt{s}})$ for $\mathtt{s} \in \hat{\mathbf{S}}$ from the reflecting Brownian motion on a closed interval (resp. the Brownian motion on $\mathbf{R}$) by a time change and a state space transformation. Since the infinitesimal generator of this diffusion is $\frac{d}{2dx}\frac{d}{d\mathsf{s}}$ in Feller's canonical form, such a construction is well known in principle (cf. [27]), but we shall formulate it in relation to the transformations of Dirichlet forms in order to ensure the recurrence of the resulting diffusion and Dirichlet form.

In the last section, we shall state some useful descriptions of the space $\mathbf{S}$ and give examples of $\mathsf{s} \in \mathbf{S} \setminus \hat{\mathbf{S}}$ corresponding to transient regular Dirichlet subspaces of $(H^1(\mathbf{R}), \frac{1}{2}\mathbf{D})$.

We recall (cf. [14, p.55]) that the extended Dirichlet space $H^1_e(I)$ of $H^1(I)$ is given by

$$H^1_e(I) = \{u : u \text{ is absolutely continuous on } I \text{ and } u' \in L^2(I)\}. \tag{5.4.7}$$

In particular, $1 \in H^1_e(I)$ and the Dirichlet form $(\frac{1}{2}\mathbf{D}, H^1(I))$ is recurrent. $H^1_e(I)$ is continuously imbedded into $C(\bar{I})$ and in fact the following elementary inequality holds for any $x, y \in \bar{I}$:

$$|u(y) - u(x)|^2 \leq |y - x|\mathbf{D}(u, u), \ u \in H^1_e(I). \tag{5.4.8}$$

When I is finite, $H^1_e(I) = H^1(I)$.

Let $(\mathscr{F}, \mathscr{E})$ be a regular Dirichlet subspace of $(H^1(I), \frac{1}{2}\mathbf{D})$. Since $(\mathscr{F}, \mathscr{E})$ is strongly local, there exists a diffusion process $\mathbf{M} = (X_t, \mathsf{P}^x)$ on $\bar{I}$ associated with it. Denote by σ_y the hitting time of the one point set $\{y\}$, $y \in \bar{I}$, for $\mathbf{M}$. The next lemma about the existence of the scale function (a strictly increasing continuous function satisfying (5.4.9)) is well known for a more general one-dimensional diffusion process ([26]) but we give a self contained proof of it based on the inequality (5.4.8) in the present special situation.

Lemma 5.4.1 There exists a strictly increasing function s on $\bar{I}$ uniquely up to a linear transformation such that

$$\mathsf{P}^x(\sigma_d < \sigma_c) = \frac{\mathsf{s}(x) - \mathsf{s}(c)}{\mathsf{s}(d) - \mathsf{s}(c)}, \quad c \leq x \leq d, \quad c, d \in \bar{I}. \tag{5.4.9}$$

s is absolutely continuous on I.

Proof: Let J be a connected open subset of $\bar{I}$. We denote by τ_J the leaving time from J of the diffusion $\mathbf{M}$. We also consider the part $\mathbf{M}_J$ of $\mathbf{M}$ on J the diffusion killed upon the leaving time τ_J. $\mathbf{M}_J$ is then associated with the subspace $\mathscr{F}_J$ of $(\mathscr{F}, \mathscr{E})$ defined by

$$\mathscr{F}_J = \{u \in \mathscr{F} : u(x) = 0,\ x \in \bar{I} \setminus J\}.$$

(5.4.8) implies that each singleton of J has a positive capacity with respect to the Dirichlet form $(\mathscr{F}_J, \mathscr{E})$. Consequently, the connectedness of the state space J is a synonym for its quasi-connectedness for $(\mathscr{F}_J, \mathscr{E})$ and hence $(\mathscr{F}_J, \mathscr{E})$ is irreducible ([14, p.172]). This implies, by virtue of [14, Theorem 4.6.6], that

$$\mathsf{P}^x(\sigma_y < \tau_J) > 0 \qquad \forall x, y \in J. \tag{5.4.10}$$

For any $c, d \in \bar{I}$, $-\infty < c < d < \infty$, we make the following choice of the intervals $J \subset \bar{I}$: when $\bar{I} = [a, b]$ (resp. $\bar{I} = \mathbf{R}$), we take $[a, d)$ and $(c, b]$ (resp. $(-\infty, d)$ and (c, ∞)). We then get from (5.4.10)

$$\mathsf{P}^x(\sigma_c < \sigma_d) > 0, \qquad \mathsf{P}^x(\sigma_d < \sigma_c) > 0, \quad \forall x \in (c, d).$$

We also note here that

$$\mathsf{P}^c(\sigma_c < \sigma_d) = 1 \qquad \mathsf{P}^d(\sigma_d < \sigma_c) = 1 \tag{5.4.11}$$

because the positivity of the capacity of a point implies the $\mathbf{M}$-regularity of the point for itself.

On the other hand, for the finite open interval $J = (c, d) \subset I$, the space $(\mathscr{F}_J, \mathscr{E})$ admits a 0-order potential operator G^0 by virtue of (5.4.8) again: for any $f \in L^2(J)$,

$$G^0 f \in \mathscr{F}_J, \qquad \mathscr{E}_J(G^0 f, v) = \int_J f v dx, \quad \forall v \in \mathscr{F}_J.$$

Therefore

$$E_x(\sigma_c \wedge \sigma_d) = G^0 1_J(x) < \infty, \quad x \in (c, d),$$

and

$$\mathsf{P}^x(\sigma_c < \sigma_d) + \mathsf{P}^x(\sigma_d < \sigma_c) = 1, \quad x \in (c,d).$$

In particular, the function $p_{c,d}(x) = \mathsf{P}^x(\sigma_d < \sigma_c),\ x \in \bar{I}$, is not only strictly positive but also strictly increasing in $x \in (c,d)$ because the sample path continuity and the strong Markov property of $\mathbf{M}$ implies

$$p_{c,d}(x) = p_{c,y}(x)p_{c,d}(y) < p_{c,d}(y), \qquad c < x < y < d. \tag{5.4.12}$$

In the same way, we have, for $c' \le c < d \le d',\ c', d' \in \bar{I}$, that

$$\begin{aligned} p_{c',d'}(x) &= p_{c,d}(x)p_{c',d'}(d) + (1 - p_{c,d}(x))p_{c',d'}(c) \\ &= (p_{c',d'}(d) - p_{c',d'}(c))p_{c,d}(x) + p_{c',d'}(c) \quad c \le x \le d. \end{aligned} \tag{5.4.13}$$

When $I = (a,b)$, we let

$$\mathtt{s}(x) = p_{a,b}(x) \qquad x \in \bar{I}.$$

Then $\mathtt{s}$ is strictly increasing and its property (5.4.9) follows from (5.4.13) with $c' = a,\ d' = b$. When $I = \mathbf{R}$, we put, for any $c < d$ such that $c \le x \le d$ and $c < 0,\ 1 < d$,

$$\mathtt{s}(x) = \alpha p_{c,d}(x) + \beta,$$

and determines constants α, β by

$$\mathtt{s}(0) = 0, \qquad \mathtt{s}(1) = 1.$$

Then, $\mathtt{s}(x)$ is independent of such a choice of (c,d) because, for any interval $(c', d') \supset (c,d)$, $p_{c,d}$ is a linear function of $p_{c',d'}$ on $[c,d]$ in view of (5.4.10). Further $\mathtt{s}$ satisfies (5.4.9) because $p_{c,d}(c) = 0,\ p_{c,d}(d) = 1$.

Finally, in order to show the absolute continuity of $\mathtt{s}$, we take any finite interval $(c,d) \subset I$. It suffices to prove that the function $p(x) = p_{c,d}(x), x \in I$, is absolutely continuous since $\mathtt{s}$ is a linear function of p on (c,d).

When $I = (a,b)$, $p(x)$ is known to be the 0-order equilibrium potential of $\{d\}$ with respect to the Dirichlet space

$$\mathscr{F}_{(c,b]} = \{u \in \mathscr{F} : u(x) = 0, \quad \forall x \le c\},$$

and $p(x)$ is characterized by

$$p \in \mathscr{F}_{(c,b]}, \quad p(d) = 1, \quad \mathscr{E}(p, v) \geq 0, \ \forall v \in \mathscr{F}_{(c,b]}, \ v(d) \geq 0. \tag{5.4.14}$$

In particular, p is absolutely continuous.

When $I = \mathbf{R}$, we consider the space

$$\mathscr{F}_{(c,\infty)} = \{u \in \mathscr{F} : u(x) = 0, \quad \forall x \leq c\}.$$

By virtue of (5.4.7), we see that the Dirichlet space $(\mathscr{F}_{(c,\infty)}, \mathscr{E})$ is transient and the function $p(x)$ is the associated 0-order equilibrium potential of $\{d\}$ characterized by

$$p \in \mathscr{F}_{(c,\infty),e}, \quad p(d) = 1, \quad \mathscr{E}(p, v) \geq 0, \ \forall v \in \mathscr{F}_{(c,\infty),e}, \ v(d) \geq 0, \tag{5.4.15}$$

where $\mathscr{F}_{(c,\infty),e} (\subset H^1_e(\mathbf{R}))$ is the extended Dirichlet space of $\mathscr{F}_{(c,\infty)}$. Hence p is absolutely continuous. □

We call the function s in Lemma 5.4.1 the *scale function* associated with the regular Dirichlet subspace $(\mathscr{F}, \mathscr{E})$ of $(H^1(I), \frac{1}{2}\mathbf{D})$.

We continue to consider a finite open interval $J = (c, d) \subset I$ and the corresponding function $p(x) = p_{c,d}(x)$ as in the proof of Lemma 5.4.1. By virtue of (5.4.8), the space $(\mathscr{F}_J, \mathscr{E})$ admits the reproducing kernel $g^0(x, y), \ x, y \in J$ characterized by

$$g^0(\cdot, y) \in \mathscr{F}_J, \quad \mathscr{E}(g^0(\cdot, y), v) = v(y), \ \forall v \in \mathscr{F}_J.$$

Lemma 5.4.2 There exists a constant $C > 0$, such that, for any $x, y \in J$,

$$g^0(x, y) = \begin{cases} Cp(x)(1 - p(y)), & x \leq y; \\ C(1 - p(x))p(y), & x \geq y. \end{cases}$$

Proof: We consider the function

$$p^0_y(x) := \mathsf{P}^x(\sigma_c \wedge \sigma_d > \sigma_y), \ x, y \in J, \tag{5.4.16}$$

$p_y^0(\cdot)$ is the 0-order equilibrium potential of $\{y\}$ with respect to $(\mathscr{F}_J, \mathscr{E})$ characterized by

$$p_y^0 \in \mathscr{F}_J,\ p_y^0(y) = 1,\ \mathscr{E}(p_y^0, v) \geq 0,\ \forall v \in \mathscr{F}_J,\ v(y) \geq 0. \tag{5.4.17}$$

The above two characterizations lead us to

$$p_y^0(x) = \frac{g^0(x,y)}{g^0(y,y)} \qquad x, y \in J.$$

On the other hand, we have $p_y^0(x) = p_{c,y}(x),\ c < x \leq y$, and we get from (5.4.13)

$$p_y^0(x) = \begin{cases} \dfrac{p(x)}{p(y)}, & x \leq y \\ \dfrac{1-p(x)}{1-p(y)}, & x \geq y, \end{cases}$$

for $x, y \in J$. The desired expression of $g^0(x,y)$ follows from the above two identities. □

Lemma 5.4.3 Any function in $\mathscr{F}$ is absolutely continuous with respect to $d\mathtt{s}$.

Proof: For any finite interval $J = (c,d) \subset I$, let G^0 be the 0-order potential operator associated with $(\mathscr{F}_J, \mathscr{E})$ as was considered in the proof of Lemma 5.4.1. Then it follows from Lemma 5.4.2 that, for $f \in L^2(J)$, $x \in J$,

$$\begin{aligned} G^0 f(x) &= \int_J g^0(x,y) f(y) dy \\ &= C(1-p(x)) \int_c^x p(y) f(y) dy + C p(x) \int_x^d (1-p(y)) f(y) dy \\ &= C p(x) \int_c^d (1-p(y)) f(y) dy - C \int_c^x \int_c^y f(z) dz dp(y), \end{aligned}$$

which means that $G^0 f$ is absolutely continuous with respect to p, namely, it can be expressed as $\int_c^x \varphi(y) p'(y) dy$ by some function $\varphi \in L^1(J; dp)$.

Since $G^0(L^2(J))$ is dense in $\mathscr{F}_J$, there exist, for any $u \in \mathscr{F}_J$, $f_n \in L^2(J)$ such that $u_n = G^0 f_n = \int_0^{\cdot} \varphi_n(y) p'(y) dy$ is $\mathscr{E}$ convergent to u. Hence, for any $B \subset J$ on which $p'(x) = 0$ a.e.,

$$\int_B u'(x)^2 dx = \int_B (u'(x) - \varphi_n(x)p'(x))^2 dx \le \mathscr{E}(u-u_n, u-u_n) \to 0, \quad n \to \infty,$$

which implies $u'(x) = 0$ a.e. on B, namely, u is absolutely continuous with respect to dp.

Finally, any $u \in \mathscr{F}$ can be expressed as

(5.4.18)

$$u(x) = u(c) + (u(d) - u(c))p(x) + [(u(x) - u(c)) - (u(d) - u(c))p(x)], \ x \in J,$$

the last term being a member of $\mathscr{F}_J$. Therefore u is absolutely continuous on J with respect to dp and hence with respect to $d\mathsf{s}$. □

Suppose that $(\mathscr{F}, \mathscr{E})$ is recurrent. Then, by [14, Theorem 4.6], the property (5.4.10) for $J = \bar{I}$ is strengthened to

(5.4.19) $$\mathsf{P}^x(\sigma_y < \infty) = 1 \qquad \forall x, y \in \bar{I}.$$

Note that, when $I = (a, b)$, $(\mathscr{F}, \mathscr{E})$ is automatically recurrent because, owing to the regularity, $\mathscr{F}$ contains a continuous function v greater than 1 on $[a, b]$ and hence the constant function $1 \wedge v$ as well.

For the scale function s associated with $(\mathscr{F}, \mathscr{E})$, we let

(5.4.20) $$E_{\mathsf{s}} = \{x \in I : \limsup_{h \to \infty} \frac{\mathsf{s}(x+h) - \mathsf{s}(x)}{h} = 0\}.$$

Lemma 5.4.4 Suppose $(\mathscr{F}, \mathscr{E})$ is recurrent.
(i) s' is constant a.e. on $I \setminus E_{\mathsf{s}}$.
(ii) $\mathsf{s}(\pm\infty) = \pm\infty$.

Proof: (i) Again we fix an arbitrary interval $(c, d) \subset I$ and denote by $p(x),\ x \in I$, the function $p_{c,d}(x)$ in the proof of Lemma 5.4.1. We know that p is absolutely continuous on I, strictly increasing on (c, d) and $p((c, d)) = (0, 1)$. Denote by q the inverse function of $p|_{(c,d)}$.

We have then

(5.4.21) $$|p(A)| = \int_A p'(x) dx, \quad \text{for any Borel set } A \subset (c, d).$$

This is clear for a disjoint union of finite number of subintervals of (c,d), and the monotone class lemma (cf.[6]) then applies.

We next let

$$E = \{x \in (c,d) : \limsup_{h \to 0} \frac{p(x+h) - p(x)}{h} = 0\}$$

and $F = p(E)$. Then $|F| = 0$ by (5.4.21). (5.4.21) further means that, if $A \subset (c,d) \setminus E$ and $|A| > 0$, then $|p(A)| > 0$. Hence q is absolutely continuous on $(0,1) \setminus F$.

On the other hand, for any $\varphi \in C_0^{(1)}((0,1))$, $\varphi(p) \in \mathscr{F}_{(c,b]}$ (resp. $\mathscr{F}_{(c,\infty),e}$) when $I = (a,b)$ (resp. $\mathbf{R}$.) Further $\varphi(p(x)) = 0,\ x \geq d, x \in I$, because $p(x) = 1,\ x \geq d, x \in I$, on account of (5.4.19) and (5.4.11). Hence, in view of (5.4.14) and (5.4.15),

$$\int_0^1 p'(q(x))\varphi'(x)dx = \int_c^d p'(x)\varphi'(p(x))p'(x)dx = \mathscr{E}(p, \varphi(p)) = 0.$$

It follows that $p'(q(x))$ is constant a.e. on $(0,1)$. Therefore p' is constant a.e. on $(c,d) \setminus E$. Since s is a linear function of p on (c,d), s' is constant a.e. on $(c,d) \setminus E_{\mathsf{s}}$ as was to be proved.
(ii) Since the recurrence assumption implies the conservativeness of the process $\mathbf{M}$ ([14]), it is easy to see that

$$\mathsf{P}^x(\lim_{y \to \pm\infty} \sigma_y = \infty) = 1 \qquad x \in \bar{I},$$

and we can get $\mathsf{s}(-\infty) = -\infty$ by noting (5.4.17) and letting $c \to -\infty$ in (5.4.9). Similarly we get $\mathsf{s}(\infty) = \infty$. $\square$

We are now in a position to state a main theorem of this paper. Let $\mathbf{S}$ be the class of functions s defined by (5.4.4) and $\hat{\mathbf{S}}$ be its subclass defined by (5.4.5). For $\mathsf{s} \in \mathbf{S}$, we introduce the space $(\mathscr{F}^{(\mathsf{s})}, \mathscr{E}^{(\mathsf{s})})$ by (5.4.2) and (5.4.3).

Theorem 5.4.1 (i) For any $\mathsf{s} \in \mathbf{S}$, the space $(\mathscr{F}^{(\mathsf{s})}, \mathscr{E}^{(\mathsf{s})})$ is a regular Dirichlet subspace of $(H^1(I), \frac{1}{2}\mathbf{D})$. The scale function associated with $(\mathscr{F}^{(\mathsf{s})}, \mathscr{E}^{(\mathsf{s})})$ equals s up to a linear transform.

(ii) Let $(\mathscr{F}, \mathscr{E})$ be a regular recurrent Dirichlet subspace of $(H^1(I), \frac{1}{2}\mathbf{D})$ and $\mathtt{s}$ be the associated scale function. Then, by making a linear modification of $\mathtt{s}$ if necessary, $\mathtt{s}$ belongs to the class $\hat{\mathbf{S}}$ and

$$\mathscr{F} = \mathscr{F}^{(\mathtt{s})}, \qquad \mathscr{E}(u,v) = \mathscr{E}^{(\mathtt{s})}(u,v),\ u,v \in \mathscr{F}.$$

Remark. The converse to (ii) (the recurrence of the space $(\mathscr{F}^{(\mathtt{s})}, \mathscr{E}^{(\mathtt{s})})$ for $\mathtt{s} \in \hat{\mathbf{S}}$) will be shown in the next section.

Proof: (i) Suppose $\mathtt{s} \in \mathbf{S}$ and $u, v \in \mathscr{F}^{(\mathtt{s})}$. Then u, v are absolutely continuous with respect to dx and

$$\text{(5.4.22)} \qquad \begin{aligned} \frac{1}{2}\int_I \frac{du}{dx}\frac{dv}{dx}dx &= \frac{1}{2}\int_I \frac{du}{d\mathtt{s}}\frac{dv}{d\mathtt{s}}\mathtt{s}'(x)^2 dx \\ &= \frac{1}{2}\int_I \frac{du}{d\mathtt{s}}\frac{dv}{d\mathtt{s}}d\mathtt{s}, \quad u,v \in \mathscr{F}^{(\mathtt{s})}. \end{aligned}$$

Hence $\mathscr{F}^{(\mathtt{s})} \subset H^1(I)$ and $\mathscr{E}^{(\mathtt{s})}(u,v) = \frac{1}{2}\mathbf{D}(u,v),\ u,v \in \mathscr{F}^{(\mathtt{s})}$.

Since $u(d) - u(c) = \int_c^d \frac{du}{d\mathtt{s}} d\mathtt{s}$, we see that

$$|u(d) - u(c)|^2 \leq 2|d-c|\mathscr{E}^{(\mathtt{s})}(u,u) \quad (c,d) \subset I, \quad u \in \mathscr{F}^{(\mathtt{s})},$$

and any $\mathscr{E}_1^{(\mathtt{s})}$-Cauchy sequence is uniformly convergent on any compact interval of I. Hence $(\mathscr{F}^{(\mathtt{s})}, \mathscr{E}^{(\mathtt{s})})$ is a closed symmetric form on $L^2(I, dx)$. Clearly it is Markovian.

The regularity is also verifiable. When I is a finite interval, $\mathscr{F}^{(\mathtt{s})}$ contains $\mathtt{s}$ and constant functions and hence an algebra generated by them, which separates points of $\bar{I}$. Consequently $\mathscr{F}^{(\mathtt{s})}$ is dense in $C(\bar{I})$ by the Weierstrass theorem. Since the above inequality implies that $\mathscr{F}^{(\mathtt{s})} \subset C(\bar{I})$, we see that $(\mathscr{F}^{(\mathtt{s})}, \mathscr{E}^{(\mathtt{s})})$ is a regular Dirichlet form on $L^2(\bar{I})$.

When $I = \mathbf{R}$, we consider the space

$$\mathscr{C} = \{\varphi(\mathtt{s}) : \varphi \in C_0^1(\mathbf{R})\}.$$

Then $\mathscr{C} \subset \mathscr{F}^{(\mathtt{s})}$. Since $\mathscr{C}$ is an algebra separating points of I, it is dense in $C_0(\mathbf{R})$. Suppose $u \in \mathscr{F}^{(\mathtt{s})}$ is $\mathscr{E}_1$-orthogonal to $\mathscr{C}$: $\mathscr{E}_1(u,v) = 0\ \forall v \in \mathscr{C}$. Then

u is a solution of the equation

$$\frac{1}{2}\frac{d}{dx}\frac{du}{d\mathtt{s}} = u.$$

It is known that the solutions of this equation form a 2-dimensional vector space spanned by a positive increasing function $u^{(1)}$ and a positive decreaing function $u^{(2)}$ ([27]). Obviously, neither $u^{(1)}$ nor $u^{(2)}$ is in $L^2(\mathbf{R})$ and u must vanish. Hence $\mathscr{C}$ is dense in $\mathscr{F}^{(\mathtt{s})}$.
Therefore $(\mathscr{F}^{(\mathtt{s})}, \mathscr{E}^{(\mathtt{s})})$ is a regular Dirichlet subspace of $(H^1(I), \frac{1}{2}\mathbf{D})$.

In order to prove the second assertion in (i), we consider any finite interval $J = (c, d) \subset \mathbf{R}$, take any $d_1 \in J$ and put

$$r(x) = \left(\frac{\mathtt{s}(x) - \mathtt{s}(c)}{\mathtt{s}(d_1) - \mathtt{s}(c)}\right)^+ \wedge \left(\frac{\mathtt{s}(d) - \mathtt{s}(x)}{\mathtt{s}(d) - \mathtt{s}(d_1)}\right)^+, \quad x \in \mathbf{R}.$$

We readily see that $r \in \mathscr{F}_J^{(\mathtt{s})}$, $r(d_1) = 1$ and, for any $v \in \mathscr{F}_J^{(\mathtt{s})}$,

$$\begin{aligned}\mathscr{E}^{(\mathtt{s})}(r, v) &= \frac{1}{2(\mathtt{s}(d_1) - \mathtt{s}(c))}\int_c^{d_1} \frac{dv}{d\mathtt{s}} d\mathtt{s} - \frac{1}{2(\mathtt{s}(d) - \mathtt{s}(d_1))}\int_{d_1}^{d} \frac{dv}{d\mathtt{s}} d\mathtt{s} \\ &= \frac{1}{2(\mathtt{s}(d_1) - \mathtt{s}(c))} v(d_1) + \frac{1}{2(\mathtt{s}(d) - \mathtt{s}(d_1))} v(d_1).\end{aligned}$$

Hence r satisfies condition (5.4.17) for $(\mathscr{F}^{(\mathtt{s})}, \mathscr{E}^{(\mathtt{s})})$ and $r(x)$ coincides with the function $p^0_{d_1}(x)$ defined by (5.4.16) on J for the diffusion (X_t, P^x) associated with $(\mathscr{F}^{(\mathtt{s})}, \mathscr{E}^{(\mathtt{s})})$ and in particular

$$r(x) = \mathsf{P}^x(\sigma_{d_1} < \sigma_c) \qquad x \in (c, d_1),$$

Since

$$r(x) = \frac{\mathtt{s}(x) - \mathtt{s}(c)}{\mathtt{s}(d_1) - \mathtt{s}(c)} \quad c < x < d_1,$$

we have shown that $\mathtt{s}$ is a scale function for the space $(\mathscr{F}^{(\mathtt{s})}, \mathscr{E}^{(\mathtt{s})})$.

(ii) The scale function $\mathtt{s}$ associated with a given regular recurrent Dirichlet subspace $(\mathscr{F}, \mathscr{E})$ of $(H^1(I), \frac{1}{2}\mathbf{D})$ belongs to $\hat{\mathbf{S}}$ (after an appropriate linear transform) by virtue of Lemma 5.4.1 and Lemma 5.4.4. We further see from Lemma 5.4.3 and identity (5.4.22) for $u, v \in \mathscr{F}$ that $\mathscr{F} \subset \mathscr{F}^{(\mathtt{s})}$ and $\mathscr{E}(u, v) = \mathscr{E}^{(\mathtt{s})}(u, v), \quad u, v \in \mathscr{F}$.

Take an interval $J = (c, d) \subset I$. Consider any function $u \in \mathscr{F}^{(\mathtt{s})}$ with $u(x) = 0$ for $x \notin J$ and assume that u is $\mathscr{E}^{(\mathtt{s})}$-orthogonal to the space $\mathscr{F}_J$:

$$\mathscr{E}(u, v) = 0, \qquad \forall v \in \mathscr{F}_J.$$

By the function $p = p_{c,d}$ as in the proof of Lemma 5.4.1, we may write

$$\mathtt{s}(x) = c_0 p(x) + c_1, \quad u(x) = \int_c^x \varphi(\xi) dp(\xi), \quad c \leq x \leq d.$$

Choosing as v the Green function $g^{0,y}(x) = g^0(x, y) \in \mathscr{F}_J$ of Lemma 5.4.2 for each fixed $y \in J$, we are led to

$$\begin{aligned} \mathscr{E}^{(\mathtt{s})}(u, g^{0,y}) &= \int_0^d \frac{du}{ds} \frac{dg^{0,y}}{ds} ds \\ &= Cc_0^{-1} \int_c^y \varphi(x)(1 - p(y)) dp(x) - Cc_0^{-1} \int_y^d \varphi(x) p(y) dp(x) \\ &= Cc_0^{-1} \int_c^y \varphi(x) dp(x) - Cc_0^{-1} p(y) \int_c^d \varphi(x) dp(x) = Cc_0^{-1} u(y), \end{aligned}$$

and $u = 0$. Hence any function in $\mathscr{F}^{(\mathtt{s})}$ with compact support belongs to the space $\mathscr{F}$. Since we have seen in (i) that $(\mathscr{E}^{(\mathtt{s})}, \mathscr{F}^{(\mathtt{s})})$ is regular, we have the desired inclusion $\mathscr{F}^{(\mathtt{s})} \subset \mathscr{F}$. □

5.5 Constructions by time change and state space transform

If the scale function of the diffusion associated with the regular Dirichlet subspace is $\mathtt{s}$, then we know intuitively that, after a state space transformation $\mathtt{s} : I \to \mathtt{s}(I)$, the diffusion becomes another diffusion with hitting distributions identical with that of Brownian motion and that this new diffusion differs from the Brownian motion by a time change. This suggests a way of constructing the original diffusion and Dirichlet subspace from the Brownian motion and Sobolev space.

In this section, we construct a recurrent diffusion process $\tilde{X}$ associated with the Dirichlet form (5.4.2),(5.4.3) on $L^2(I; dx)$ for $\mathtt{s} \in \hat{\mathbf{S}}$ from the reflecting Brownian motion on $\mathtt{s}(\bar{I})$ when I is finite and the Brownian motion on $\mathbf{R}$

when $I = \mathbf{R}$ by a time change and a transformation of the state space. We also notice that $\tilde{X}$ is the one-dimensional diffusion on $\bar{I}$ with infinitesimal generator $\frac{1}{2}\frac{d}{dx}\cdot\frac{d}{d\mathtt{s}}$ in Feller's sense ([26]).

We prepare a lemma.

Lemma 5.5.1 Let (E, m) be a σ-finite measure space, $X = (X_t, \mathsf{P}^x)$ be an m-symmetric Markov process on E and $(\mathscr{F}, \mathscr{E})$ be the associated Dirichlet space on $L^2(E; m)$. Let γ be a one-to-one measurable transformation from E onto a space $\tilde{E}$ and $\tilde{m}$ be the image measure; $\tilde{m}(B) = m(\gamma^{-1}(B))$. We put

$$\tilde{X}_t = \gamma(X_t), \qquad \tilde{P}_x = P_{\gamma^{-1}x},\ x \in \tilde{E}.$$

Then $\tilde{X} = (\tilde{X}_t, \tilde{P}_x)$ is an $\tilde{m}$-symmetric Markov process on $\tilde{E}$ and the associated Dirichlet space $(\tilde{\mathscr{F}}, \tilde{\mathscr{E}})$ on $L^2(\tilde{E}, \tilde{m})$ satisfies

$$\tilde{\mathscr{F}} = \{u \in L^2(\tilde{E}; \tilde{m}) : u\circ\gamma \in \mathscr{F}\}$$

$$\tilde{\mathscr{E}}(u, v) = \mathscr{E}(u\circ\gamma, v\circ\gamma) \qquad u, v \in \tilde{\mathscr{F}}.$$

Proof: It was proved in p325 [6] that $\tilde{X}$ is a Markov process on $\tilde{E}$ with transition function

$$\tilde{p}_t f(y) = p_t(f\circ\gamma)(\gamma^{-1}(y)) \quad y \in \tilde{E},\ f \in \mathscr{B}^+,$$

where p_t is the transition function of X.

The $\tilde{m}$-symmetry of $\tilde{p}_t$ and the above relation of the Dirichlet spaces follow from

$$\begin{aligned}\int_{\tilde{E}} \tilde{p}_t f \cdot g d\tilde{m} &= \int_{\tilde{E}} p_t(f\circ\gamma)(\gamma^{-1}(y))(g\circ\gamma)(\gamma^{-1}(y))dm(\gamma^{-1}y) \\ &= \int_E p_t(f\circ\gamma)(g\circ\gamma)dm,\end{aligned}$$

and

$$\frac{1}{t}\int_{\tilde{E}}(f - \tilde{p}_t f)\cdot g d\tilde{m} = \frac{1}{t}\int_E (f\circ\gamma - p_t(f\circ\gamma))g\circ\gamma dm.$$

That completes the proof. □

The process $\tilde{X} = (\tilde{X}_t, \tilde{P}_x)_{x\in\tilde{E}}$ in the above lemma is called the process obtained from $X = (X_t, \mathsf{P}^x)_{x\in E}$ by the transformation γ of the state space from E to $\tilde{E}$.

Take any $\mathtt{s}$ from the class $\hat{\mathbf{S}}$ defined by (5.4.5) and let $\mathtt{t}$ be its inverse function. Clearly

$$J = \mathtt{s}(I) = \begin{cases} (0, b-a-|E_{\mathtt{s}}|), & I = (a,b), \\ \mathbf{R}, & I = \mathbf{R}. \end{cases}$$

Let $(B_t, \mathsf{P}^x)_{x\in\bar{J}}$ be the reflecting Brownian motion $\bar{J}$ when $I = (a,b)$ and the Brownian motion on $\mathbf{R}$ when $I = \mathbf{R}$. It is associated with the regular local recurrent Dirichlet form $(\frac{1}{2}\mathbf{D}, \mathscr{H}^1(J)$ on $L^2(\bar{J}, dx)$. The transition function of (B_t, P^x) is absolutely continuous with respect to dx. Each one point set has a positive 1-capacity with respect to this Dirichlet form. Hence the quasi-support of a positive Radon measure on $\bar{J}$ coincides with its topological support.

Let A_t be the PCAF (positive continuous additive functional) in the strict sense (B_t, P^x) with Revuz measure $d\mathtt{t}$. Since the support of $\mathtt{t}$ is $\bar{J}$, the fine support of A_t is also $\bar{J}$ and A_t is strictly increasing in t a.s. Let τ_t be the inverse of A_t and denote by X the time change of B_t by (τ_t):

$$X_t = B_{\tau_t}. \tag{5.5.1}$$

Theorem 5.5.1 (i) Let

$$\tilde{X}_t = \mathtt{t}\left(B_{\tau_t}\right), \ t \geq 0, \quad \tilde{P}_x = P_{\mathtt{s}(x)}, \ x \in \bar{I}. \tag{5.5.2}$$

Then $(\tilde{X}_t, \tilde{P}_x)_{x\in\bar{I}}$ is a diffusion process on $\bar{I}$ associated with the regular Dirichlet subspace $(\mathscr{E}^{(\mathtt{s})}, \mathscr{F}^{(\mathtt{s})})$ on $L^2(\bar{I}; dx)$ of $(H^1(I), \frac{1}{2}\mathbf{D})$.
(ii) $(\mathscr{E}^{(\mathtt{s})}, \mathscr{F}^{(\mathtt{s})})$ is recurrent.

Proof: (i) By virtue of (6.2.22) in [14], the time changed process $(X_t, \mathsf{P}^x)_{x\in\bar{J}}$ is $d\mathtt{t}$-symmetric and its Dirichlet space $(\mathscr{F}^J, \mathscr{E}^J)$ on $L^2(\bar{J}; d\mathtt{t})$ is given by

$$\mathscr{F}^J = H^1_e(J) \cap L^2(\bar{J}, d\mathtt{t}), \qquad \mathscr{E}^J(u,v) = \frac{1}{2}\mathbf{D}(u,v), \ u, v \in \mathscr{F}^J,$$

for the extended Dirichlet space $H^1_e(J)$ defined by (5.4.7) for $H^1(J)$.

Since $\tilde{X} = (\tilde{X}_t, \tilde{P}_x)$ is obtained from the time changed process (X_t, P^x) of (5.5.1) by means of the transformation $\mathtt{t}$ of the state space from $\bar{J}$ onto $\bar{I}$, we see by Lemma 5.5.1 that $\tilde{X}$ is symmetric with respect to the image measure by $\mathtt{t}$ of $d\mathtt{t}$, which is obviously the Lebesgue measure dx on $\bar{I}$, and the associated Dirichlet space $(\tilde{\mathscr{F}}, \tilde{\mathscr{E}})$ on $L^2(\bar{I}; dx) = L^2(I; dx)$ is given by

$$(5.5.3)\quad \tilde{\mathscr{F}} = \{u \in L^2(I; dx) : u\circ\mathtt{t} \in \mathscr{F}_J\} = \{u \in L^2(I, dx) :\ u\circ\mathtt{t} \in H^1_e(J)\},$$

$$(5.5.4)\qquad \tilde{\mathscr{E}}(u, v) = \frac{1}{2}\mathbf{D}(u\circ\mathtt{t}, v\circ\mathtt{t}), \quad u, v \in \tilde{\mathscr{F}}.$$

We claim that

$$(5.5.5)\qquad \tilde{\mathscr{F}} = \mathscr{F}^{(\mathtt{s})}, \qquad \tilde{\mathscr{E}}(u, v) = \tilde{\mathscr{E}}^{(\mathtt{s})}(u, v),\ u, v \in \tilde{\mathscr{F}}.$$

By (5.5.3), $u \in \tilde{\mathscr{F}}$ if and only if $u \in L^2(I; dx)$ and there exists a function $\phi \in L^2(J)$ such that

$$u(\mathtt{t}(x)) = \int_0^x \phi(y)dy + C, \quad x \in J,$$

for some constant C. In this case,

$$u(x) = \int_0^{\mathtt{s}(x)} \phi(y)dy + C = \int_a^x \phi(\mathtt{s}(y))d\mathtt{s}(y) + C,\ x \in I$$

and

$$\frac{1}{2}\int_I \left(\frac{du}{d\mathtt{s}}\right)^2 d\mathtt{s} = \frac{1}{2}\int_I \phi(\mathtt{s}(x))^2 d\mathtt{s}(x) = \frac{1}{2}\int_J \phi(x)^2 dx,$$

and hence $\tilde{\mathscr{F}} \subset \mathscr{F}^{(\mathtt{s})}$ and $\tilde{\mathscr{E}} = \mathscr{E}^{(\mathtt{s})}$ on $\tilde{\mathscr{F}} \times \tilde{\mathscr{F}}$. Converse inclusion can be shown in the same way.

(ii) We have only to show this for $I = \mathbf{R}$. By virtue of [14, (6.2.23)], the extended Dirichlet space of $(\mathscr{F}^{\mathbf{R}}, \mathscr{E}^{\mathbf{R}})$ coincides with $(H^1_e(\mathbf{R}), \frac{1}{2}\mathbf{D})$ and hence contains constant functions. Since the Dirichlet space $(\tilde{\mathscr{F}}, \tilde{\mathscr{E}})$ is obtained by (5.5.3) and (5.5.4), its extended Dirichlet space also contains constant functions. □

From the proof, it also follows that ,for $\mathtt{s} \in \hat{\mathbf{S}}$, $u \in \mathscr{F}^{(\mathtt{s})}$ if and only if $u \circ \mathtt{t} \in H^1(J)$. Equivalently $\mathscr{F}^{(\mathtt{s})} = \{u \circ \mathtt{s} : u \in H^1(J)\}$.

We can give more tractable descriptions of the class $\mathbf{S}$ of scale functions defined by (5.4.4).

Let $\mathbf{T}$ be the totality of function $\mathtt{t}$ defined on some open interval $J \subset \mathbf{R}$ expressed as

$$\mathtt{t}(x) = c(x) + x, \ x \in J, \tag{5.5.6}$$

for a non-decreasing singular continuous function $c(x)$ on J.

Let $\mathbf{E}$ be the totality of measurable subset E of I satisfying that, for any $x, y \in I, \ x < y$, $|(I \setminus E) \cap (x, y)| > 0$, i.e., the complement of E has a positive measure on any non-empty open subinterval. Two sets in $\mathbf{E}$ are regarded to be equivalent if they differ by a zero-measure set.

The following theorem illustrates the structure of $\mathbf{S}$ and shows that any regular recurrent Dirichlet subspace of $(H^1(I), \mathbf{D})$ may be obtained in the same way as done in the example in §5.4.

Theorem 5.5.2 Let $\mathtt{s}$ be a strictly increasing function on I.

1. $\mathtt{s} \in \mathbf{S}$ if and only if its inverse function belongs to $\mathbf{T}$.

2. $\mathtt{s} \in \mathbf{S}$ if and only if there exists a set $E \in \mathbf{E}$ such that

$$\mathtt{s}(x) = \int_\eta^x 1_{E^c}(y)dy, \ x \in I, \tag{5.5.7}$$

   where η denotes a when $I = (a, b)$ and 0 when $I = \mathbf{R}$. The set E is uniquely decided by $\mathtt{s}$ up to the equivalence.

Proof. (1) For $\mathtt{s} \in \mathbf{S}$, we let $\mathtt{t}(x) = \mathtt{s}^{-1}(x), \ x \in J = \mathtt{s}(I)$. In view of the first part of the proof of Lemma 5.4.4 (i), we see that $\mathtt{t}'(x) = 1$ a.e. $x \in J$, and accordingly

$$\mathtt{t}(x) = c(x) + x, \ x \in J,$$

for some nondecreasing singular continuous function $c(x)$. Hence $\mathtt{t} \in \mathbf{T}$.

Conversely if $\mathtt{t} \in \mathbf{T}$, then $\mathtt{t}(x) = c(x) + x$ is a strictly increasing continuous function with $\mathtt{t}' = 1$ a.e. on J. Further, for any $x, y \in J, \ x < y$,,

$(y-x) \le \mathtt{t}(y) - \mathtt{t}(x)$. It follows that $\mathtt{s}(x) = \mathtt{t}^{-1}(x)$, $x \in I = \mathtt{t}(J)$, is absolutely continuous. Clearly $\mathtt{s}$ is differentiable at $\mathtt{t}(x)$ if and only if $\mathtt{t}$ has a non-zero derivative at $x \in J$ and hence

$$\mathtt{s}'(\mathtt{t}(x)) = \frac{1}{\mathtt{t}'(x)} = 1, \text{ a.e.} x \in J,$$

which implies that $\mathtt{s}' = 1$ a.e. on $I \setminus E_{\mathtt{s}}$ in the same way as in the second part of the proof Lemma 5.4.4(i).

As for (2), for any $\mathtt{s} \in \mathbf{S}$, $E_{\mathtt{s}} \in \mathbf{E}$ and conversely for $E \in \mathbf{E}$, it is easy to check that $\mathtt{s} \in \mathbf{S}$ as defined in (5.5.7). □

By this theorem, we can readily conceive functions in $\mathbf{S} \setminus \hat{\mathbf{S}}$ when $I = \mathbf{R}$. For example, for any non-decreasing singular continuous function $c(x)$ on $\mathbf{R}$ with $c(\pm\infty) = \pm\infty$, we put

$$\mathtt{t}(x) = c\left(\frac{x}{1-|x|}\right) + x, \qquad x \in (-1,1) \tag{5.5.8}$$

and let $\mathtt{s}$ be the inverse function of $\mathtt{t}$.

Another example is provided by

$$\mathtt{s}(x) = \int_0^x 1_G(y)dy, \ x \in \mathbf{R}, \text{ for } G = \bigcup_{r_n \in Q} (r_n - \frac{1}{2^{n+1}}, r_n + \frac{1}{2^{n+1}}), \tag{5.5.9}$$

where $Q = \{r_n\}$ is the set of all rational numbers.

In both cases, $\mathtt{s}(-\infty)$ and $\mathtt{s}(\infty)$ are finite and the corresponding spaces $(\mathscr{F}^{(\mathtt{s})}, \mathscr{E}^{(\mathtt{s})})$ are transient Dirichlet subspaces of $(H^1(\mathbf{R}), \frac{1}{2}\mathbf{D})$ by Theorem 5.4.1.

5.6 Dirichlet forms of linear diffusions

Due to the pioneering works of Feller, one-dimensional diffusion has been a mature and very interesting topic in theory of Markov processes with its simplicity and clarity. There are a lot of literatures on this topic, e.g., Ito-Mckean[27], Revuz-Yor[31], Rogers-Williams [36], among those most influential. As we shall see, one-dimensional irreducible diffusion is always symmetric. Thus it has no loss of generality that Dirichlet form approach is introduced to investigate one-dimensional diffusions. In this article, we shall discuss the properties of Dirichlet spaces associated with one-dimensional diffusions, and study one-dimensional diffusions by means of Dirichlet forms.

Let I be an interval or a connected subset of $\mathbf{R}$ and I° its interior.

Definition 5.6.1 A diffusion $X = (X_t, \mathsf{P}^x)$ with life time ζ on I is a Hunt process on I with continuous sample paths on $[0, \zeta)$. A diffusion X is called irreducible if for any $x, y \in I$, $\mathsf{P}^x(T_y < \infty) > 0$, where T_y denotes the hitting time of y.

The irreducibility defined here implies the regularity in [31] and [36]. The reason we use irreducibility is that I is the state space of X, while in [31] and [36], I may contain a trap, thus not a real state space. Another thing which needs to be noted is that a diffusion defined this way is allowed being 'killed' inside I, while in some literature it is not allowed. A diffusion not allowed being killed inside I is called locally conservative. The local conservativeness is equivalent to the following property: for any $x \in I^\circ$, there exist $a, b \in I$ with $a < b$ and $x \in (a, b)$ such that $\mathsf{P}^x(T_a \wedge T_b < \infty) = 1$; if x is the right (resp. left) end-point of I included in I and finite, then there exists $a \in I$ and $a < x$ (resp. $a > x$) such that $\mathsf{P}^x(T_a < \infty) = 1$. For any regular diffusion X, we shall obtain a process X' through the well-known Ikeda-Nagasawa-Watanabe piecing together procedure. It is easy to show that X' is a locally conservative regular diffusion on I, and X is obtained by killing X' at a rate given by a PCAF. We say that X' is a resurrected process of X and X is a subprocess of X'. As VII(3.2) in [31] or (46.12) in [36], a locally conservative regular diffusion X on I has so-called scale function, namely, there exists a continuous, strictly increasing function $\mathtt{s}$ on I such that for any $a, b, x \in I$ with $a < b$ and $a \le x \le b$,

$$\mathsf{P}^x(T_b < T_a) = \frac{\mathtt{s}(x) - \mathtt{s}(a)}{\mathtt{s}(b) - \mathtt{s}(a)}. \tag{5.6.1}$$

The function $\mathtt{s}$ is unique up to a linear transformation. This function $\mathtt{s}$ is called a scale function of X. A diffusion with scale function $\mathtt{s}(x) = x$ is said to be in natural scale. It is easy to check that if $\mathtt{s}$ is a scale function of X, then $\mathtt{s}(X)$ is a diffusion on $\mathtt{s}(I)$ in natural scale. A Brownian motion on I is a diffusion on I which moves like Brownian motion inside I and is reflected at any end-point which is finite and in I and get absorbed at any end point which is finite but not in I. Clearly Brownian motion on I is clearly in natural scale. Thus Blumenthal-Getoor-Mckean's theorem (Theorem 5.5.1 [3]) implies that a diffusion on I in natural scale is identical in law with a time change of Brownian motion on I. More precisely, let X be a locally conservative regular diffusion in natural scale. Then there exists a measure ξ on $\mathbf{R}$, fully supported on I, and a Brownian motion $B = (B_t)$ on I such that X is equivalent in law to (B_{τ_t}) where $\tau = (\tau_t)$ is the continuous inverse of the PCAF $A = (A_t)$ of B with Revuz measure ξ. The measure ξ is called the speed measure of X. Obviously X is symmetric with respect to ξ.

Let now X be an irreducible diffusion on I and X' the resurrected process of X with scale function $\mathtt{s}$. Then $\mathtt{s}(X')$ is symmetric with respect to its speed measure ξ and therefore X' is symmetric with respect to $\xi\circ\mathtt{s}$. The diffusion X, the subprocess of X', is certainly still symmetric to $\xi\circ\mathtt{s}$. An m-symmetric Markov process on state space E always determines a Dirichlet form on $L^2(E, m)$. A standard reference for theory of Dirichlet form is [14], to which we refer for terminologies, notations and results. By results in theory of Dirichlet form, the Dirichlet form associated with X' is strongly local, irreducible and regular on I. It follows then that the Dirichlet form associated with X is local, irreducible and regular on I. Conversely, given a local, irreducible and regular Dirichlet form on $L^2(I, m)$ with a fully supported Radon measure m on I, it is easily seen that the corresponding Markov process must be an irreducible diffusion on I. Therefore one-dimensional irreducible diffusions are in one-to-one correspondence with one-dimensional local, irreducible and regular Dirichlet forms. This illustrates that no generality will be lost if we start from such a Dirichlet form as we shall do in the following sections.

In this section, we shall present a sufficient condition for the uniqueness of symmetrizing measure. Actually, this condition is almost necessary too. Then we will give a representation for any 1-dim local, irreducible and regular

Dirichlet space. Finally, we will give a necessary and sufficient condition for a Dirichlet space to be regular subspace of another Dirichlet space, which generalizes the main result in [7]. As application, two examples is presented to illustrate that Brownian motion has not only regular extensions and but also non-conservative regular subspaces.

We first present a theorem which states a condition for uniqueness of symmetrizing measure and will be used later. This kind of results may be known in some other forms. We begin with a general right Markov process $X = (X_t, \mathsf{P}^x)$ on state space E with semigroup (P_t) and resolvent (U^α). It is easy to see from the right continuity that for $x \in E$ and a finely open subset D, $\mathsf{P}^x(T_D < \infty) > 0$ if and only if $U^\alpha 1_D(x) > 0$. The process X is called irreducible if $\mathsf{P}^x(T_D < \infty) > 0$ for any $x \in E$ and a finely open subset D, where T_D is the hitting time of D.

Lemma 5.6.1 The following statements are equivalent.

1. X is irreducible.
2. $U^\alpha 1_D$ is positive everywhere on E for any non-empty finely open set D.
3. $U^\alpha 1_A$ is either identically zero or positive everywhere on E for any Borel set A.

Proof. The equivalence of (1) and (2) is easy. We shall prove that they are equivalent to (3). We may assume $\alpha = 0$. Suppose (1) is true. If $U1_A$ is not identically zero, then there exists $\delta > 0$ such that $D := \{U1_A > \delta\}$ is non-empty. Since $U1_A$ is excessive and thus finely continuous, D is finely open and the fine closure of D is contained in $\{U1_A \geq \delta\}$. Then

$$U1_A(x) \geq P_D U1_A(x) = \mathsf{E}^x\left(U1_A(X_{T_D})\right) \geq \delta \mathsf{P}^x(T_D < \infty) > 0.$$

Conversely suppose (3) is true. Then for any finely open set D, by the right continuity of X, $U1_D(x) > 0$ for any $x \in D$. Therefore $U1_D$ is positive everywhere on E. □

A Borel set A is called of potential zero if $U^\alpha 1_A$ is identically zero for some $\alpha \geq 0$ (thus for all $\alpha \geq 0$). A σ-finite measure μ on E is said to be a symmetrizing measure of X or X is said to be μ-symmetric if

$$(P_t u, v)_\mu = (u, P_t v)_\mu$$

for any measurable $u, v \geq 0$ and $t > 0$. It is easy to check that any symmetrizing measure is excessive and an excessive measure does not charge any set of potential zero.

Theorem 5.6.1 Assume that X is irreducible. Then the symmetrizing measure of X is unique up to a constant. More precisely if both μ and ν are non-trivial symmetrizing measures of X, then $\nu = c\mu$ with a positive constant c.

Proof. We may assume that there exists a measurable set H such that both $\mu(H)$ and $\nu(H)$ are positive and finite. Indeed, since both measures are non-trivial and σ-finite, we may find a set H such that $0 < \mu(H) < \infty$ and $\nu(H) < \infty$. If $\nu(H) = 0$, we may replace ν by $\nu + \mu$. Then the following argument will show that μ and $\nu + \mu$ are linear, and so are μ and ν.

Set $c = \nu(H)/\mu(H)$. We may assume that $c = 1$ without loss of generality. Let $m = \mu + \nu$. Then there is $f_1, f_2 \geq 0$ such $\mu = f_1 \cdot m$ and $\nu = f_2 \cdot m$. Let $A = \{f_1 > f_2\}$, $B = \{f_1 = f_2\}$ and $C = \{f_1 < f_2\}$.

We shall show that $\nu = \mu$. Otherwise $\mu(A) > 0$ or $\nu(C) > 0$. We assume that $\mu(A) > 0$ without loss of generality. Since μ is σ-finite, there is $A_n \in \mathscr{B}(E)$ such that $A_n \subseteq A$, $\mu(A_n) < \infty$ and $A_n \uparrow A$. Let $D = B \cup C$. For any integer n and $\alpha > 0$,

$$(U^\alpha 1_{A_n}, 1_D)_\mu \leq (U^\alpha 1_{A_n}, 1_D)_\nu = (U^\alpha 1_D, 1_{A_n})_\nu \leq (U^\alpha 1_D, 1_{A_n})_\mu.$$

Since $(U^\alpha 1_{A_n}, 1_D)_\mu = (U^\alpha 1_D, 1_{A_n})_\mu$, it follows that $(U^\alpha 1_D, 1_{A_n})_\nu = (U^\alpha 1_D, 1_{A_n})_\mu$. Thus we have

$$(U^\alpha 1_D, (1 - \frac{f_2}{f_1}) 1_{A_n})_\mu = (U^\alpha 1_D, 1_{A_n})_\mu - (U^\alpha 1_D, 1_{A_n})_\nu = 0.$$

Since $1 - \frac{f_2}{f_1} > 0$ on A, let n go to infinity and by the monotone convergence theorem we get that $(U^\alpha 1_D, 1_A)_\mu = 0$. The irreducibility of X implies that

$U^\alpha 1_D = 0$ identically or D is of potential zero. Therefore

$$\mu(D) = \nu(D) = 0.$$

Consequently,

$$0 = \mu(H) - \nu(H) = \int_{H\cap A} (1 - \frac{f_2}{f_1}) d\mu$$

which leads to that $\mu(H \cap A) = 0$ and also $\mu(H) = 0$. The contradiction implies that $\nu = \mu$. □

The following example shows that the condition that any point may reach any finely open set is needed. Actually we may easily see that it is also necessary in the sense that if X has a unique symmetrizing measure m, then X, restricted on the fine support of m, is irreducible.

Example: Let $J = \frac{1}{4}(\delta_1 + \delta_{-1} + \delta_{\sqrt{2}} + \delta_{-\sqrt{2}})$ defined on $\mathbb{R}$ and $\pi = \{\pi_t\}_{t>0}$ the corresponding symmetric convolution semigroup; i.e., $\hat{\pi}_t(x) = e^{-t\phi(x)}$ with

$$\phi(x) = \int (1 - \cos xy) J(dy) = \frac{1}{2}(1 - \cos x) + \frac{1}{2}(1 - \cos \sqrt{2}x).$$

Let $N = \{n + m\sqrt{2} \ : \ n, m \text{ are integers}\}$ and $\mu = \sum_{x\in N} \delta_x$. Then μ is σ-finite and also a symmetrizing measure. It is easy to check that any point may reach any open set but not any finely open set.

It is known that the fine topology is determined by the process and hard to identify usually. Hence it is hard to verify sometimes the irreducibility defined in the theorem. However under LSC, namely, assuming that $U^\alpha 1_B$ is lower-semi-continuous for any Borel subset B of E, the irreducibility is equivalent to the weaker one, which is easier to verify: $\mathsf{P}^x(T_D < \infty) > 0$ for any $x \in E$ and open subset $D \subset E$.

Fixing an interval I and given a fully-supported Radon measure m on I, we shall consider in this section the representation of a local, irreducible and regular Dirichlet space $(\mathscr{E}, \mathscr{F})$ on $L^2(I, m)$ in terms of the scale function of the associated diffusion. The form $(\mathscr{E}, \mathscr{F})$ is assumed to be irreducible, i.e., the associated semigroup is m-invariant. Let $X = (X_t, \mathsf{P}^x)$ be the diffusion process on I associated with $(\mathscr{E},\mathscr{F})$. It is well known that the process $X=(X_t,\mathsf{P}^x)$ associated with a local irreducible regular Dirichlet space $(\mathscr{E},\mathscr{F})$ on $L^2(I,m)$ is an irreducible m-symmetric diffusion on I. In addition $(\mathscr{E},\mathscr{F})$ is strong local if and only if (X_t,P^x) is locally conservative.

Denote by $\mathscr{S}(I)$ the totality of strictly increasing continuous functions on I. Let $\mathtt{s}\in\mathscr{S}(I)$. Let m and k two Radon measures on I with $\mathrm{supp}\,(m)=I$. Define a symmetric form $(\mathscr{E}^{(\mathtt{s},m,k)},\mathscr{F}^{(\mathtt{s},m,k)})$ as follows:

$$\mathscr{F}^{(\mathtt{s},m,k)}=\{u\in L^2(I,m+k):u\ll\mathtt{s}\text{ and }\frac{du}{d\mathtt{s}}\in L^2(I,d\mathtt{s})\}$$
$$\mathscr{E}^{(\mathtt{s},m,k)}(u,v)=\int_I\frac{du}{d\mathtt{s}}\frac{dv}{d\mathtt{s}}d\mathtt{s}+\int_I u(x)v(x)k(dx),\quad\text{for }u,v\in\mathscr{F}^{(\mathtt{s},m,k)}.$$

It follows from [8] that $\mathscr{F}^{(\mathtt{s},m,k)}$ is the closure of the algebra generated by $\mathtt{s}$ with respect to the norm $\sqrt{\mathscr{E}^{(\mathtt{s},m,k)}(\cdot,\cdot)+(\cdot,\cdot)_m}$. As in [14], if $I=\langle a_1,a_2\rangle$, we call a_1 a regular boundary if $a_1\notin I$, $\mathtt{s}(a_1)>-\infty$ and $m((a_1,c))+k((a_1,c))<\infty$ for some $c\in I$. The regularity of a_2 is defined similarly. Define also

$$\mathscr{F}_0^{(\mathtt{s},m,k)}=\{u\in\mathscr{F}^{(\mathtt{s},m,k)}:u(a_i)=0\text{ if }a_i\text{ is regular boundary }\};$$
$$\mathscr{E}_0^{(\mathtt{s},m,k)}(u,v)=\mathscr{E}^{(\mathtt{s},m,k)}(u,v),\quad\text{for }u,v\in\mathscr{F}_0^{(\mathtt{s},m,k)}.$$

When $k=0$, we write it as $(\mathscr{E}_0^{(\mathtt{s},m)},\mathscr{F}_0^{(\mathtt{s},m)})$ for simplicity. The next lemma asserts that a Dirichlet form is built this way.

Lemma 5.6.2 The form $(\mathscr{E}_0^{(\mathtt{s},m,k)},\mathscr{F}_0^{(\mathtt{s},m,k)})$ is a local irreducible Dirichlet space on $L^2(I;m)$ regular on I and it is strong local if and only if $k=0$.

Proof. We only prove the first statement. The second is clear. Let $J=\mathtt{s}(I)$ and define a regular Dirichlet space $(\mathscr{E},\mathscr{F})$ on $L^2(J,m\circ\mathtt{s}^{-1})$ (refer to [14, Example 1.2.2] for a proof) as follows:

$$\mathscr{F}=\{u\in L^2(J,(m+k)\circ\mathtt{s}^{-1}):u\text{ is absolutely continuous and }u'\in L^2(J)\}$$
$$\mathscr{E}(u,v)=\int_J u'(x)v'(x)dx+\int_J u(x)v(x)(k\circ\mathtt{s}^{-1})(dx),\ \text{for }u,v\in\mathscr{F}.$$

Then $(\mathscr{E}_0^{(\mathtt{s},m,k)},\mathscr{F}_0^{(\mathtt{s},m,k)})$ is a state-space transform of $(\mathscr{E},\mathscr{F})$ induced by the function $\mathtt{s}^{-1}$. It shows that $(\mathscr{E}_0^{(\mathtt{s},m,k)},\mathscr{F}_0^{(\mathtt{s},m,k)})$ is a Dirichlet form on $L^2(I,m)$ by [7, lemma 3.1]. The regularity follows from the fact that $u\circ\mathtt{s}^{-1} \in \mathscr{F}_0^{(\mathtt{s},m,k)} \cap C_c(I)$ whenever $u \in \mathscr{F} \cap C_c(J)$. The local property of $(\mathscr{F}_0^{(\mathtt{s},m,k)}, \mathscr{E}_0^{(\mathtt{s},m,k)})$ is obvious. $\square$

Next we give the representation theorem of one-dimensional local, irreducible and regular Dirichlet space.

Theorem 5.6.2 Let $I = \langle a_1, a_2 \rangle$ be any interval and m a Radon measure on I with $\mathrm{supp}\,(m) = I$. If $(\mathscr{E}, \mathscr{F})$ be a local irreducible regular Dirichlet space on $L^2(I,m)$, then

$$(\mathscr{E}, \mathscr{F}) = (\mathscr{E}_0^{(\mathtt{s},m,k)}, \mathscr{F}_0^{(\mathtt{s},m,k)})$$

where k is a Radon measure on I and $\mathtt{s} \in \mathscr{S}(I)$. Furthermore $\mathtt{s}$ is a scale function for (X_t, P^x) which is the diffusion associated with $(\mathscr{E}, \mathscr{F})$.

Proof. We shall first assume that $(\mathscr{E}, \mathscr{F})$ is strongly local. Let $\mathtt{s}$ be a scale function of $X = (X_t, \mathsf{P}^x)$ associated with $(\mathscr{E}, \mathscr{F})$, and $Y = (Y_t, \mathbf{Q}^x), x \in I$ be the diffusion associated with Dirichlet space $(\mathscr{F}_0^{(\mathtt{s},m)}, \mathscr{E}_0^{(\mathtt{s},m)})$. Then X and Y have the same scale function and thus the same hitting distributions. It follows from Blumenthal-Getoor-Mckean Theorem that there exists a strictly increasing continuous additive functional A_t of X such that $(Y_t, \mathbf{Q}^x), x \in I$ and $(\tilde{X}_t, \mathsf{P}^x), x \in I$ are equivalent, where $\tilde{X}_t = X_{\tau_t}$, and (τ_t) is the inverse of (A_t).

Note that $(\tilde{X}_t, \mathsf{P}^x), x \in I$ is ξ-symmetric, where ξ is the Revuz measure of A with respect to m, and also m-symmetric since it is equivalent to $(Y_t, \mathbf{Q}^x), x \in I$. By Theorem 5.6.1, ξ is a multiple of m or $A_t = ct$ for some positive constant c. It shows that $\tilde{X}_t = X_{\frac{t}{c}}$. Therefore

$$\mathscr{F} = \mathscr{F}_0^{(\mathtt{s},m)},\ \mathscr{E} = c \cdot \mathscr{E}_0^{(\mathtt{s},m)}$$

by (1.3.15) and (1.3.17) in [14].

However scale functions of a linear diffusion could differ by a linear transform. When the scale function is properly chosen, the constant c above could be 1 (and shall be taken to be 1 in the sequel). For example $\mathtt{s}' = \mathtt{s}/c \in \mathscr{S}(I)$ is also a scale function for (X_t, P^x) and we have

$$\mathscr{F} = \mathscr{F}_0^{(\mathtt{s}',m)},\ \mathscr{E} = \mathscr{E}_0^{(\mathtt{s}',m)}.$$

In general, when $(\mathscr{E},\mathscr{F})$ is local, we have the following Beurling-Deny decomposition by [14, Theorem 3.2.1]

$$\mathscr{E}(u,v)=\mathscr{E}^{(c)}(u,v)+\int_I u(x)v(x)k(dx),\ u,v\in\mathscr{F}\cap C_0(I),$$

where $\mathscr{E}^c$ is the strongly local part of $\mathscr{E}$. Define a new symmetric form $(\mathscr{E}',\mathscr{F}')$ on $L^2(I,m+k)$:

$$\mathscr{F}'=\mathscr{F},\mathscr{E}'=\mathscr{E}^{(c)}.$$

Then $(\mathscr{E}',\mathscr{F}')$ is a strongly local irreducible regular Dirichlet space on $L^2(I,m+k)$. By the conclusion in the first part, it follows that

$$\mathscr{E}^{(c)}=\mathscr{E}_0^{(\mathtt{s},m)},\mathscr{F}=\mathscr{F}'=\mathscr{F}_0^{(\mathtt{s},m+k)}=\mathscr{F}_0^{(\mathtt{s},m,k)}.$$

The proof is completed. □

Let $(\mathscr{E}',\mathscr{F}')$ and $(\mathscr{E},\mathscr{F})$ be two irreducible regular Dirichlet spaces on $L^2(I,m)$. The space $(\mathscr{E}',\mathscr{F}')$ is called a regular subspace of $(\mathscr{E},\mathscr{F})$ if $\mathscr{F}'\subset\mathscr{F}$ and $\mathscr{E}(u,v)=\mathscr{E}'(u,v)$ for any $u,v\in\mathscr{F}'$. All non-trivial regular subspaces of linear Brownian motion is characterized clearly in [7]. In this section we shall further give a necessary and sufficient condition for $(\mathscr{E}',\mathscr{F}')$ to be a regular Dirichlet subspace of $(\mathscr{E},\mathscr{F})$, which extends the result in [7].

Using the representation in §3, we have

$$(\mathscr{E},\mathscr{F})=(\mathscr{E}_0^{(\mathtt{s}_1,m,k_1)},\mathscr{F}_0^{(\mathtt{s}_1,m,k_1)});$$
$$(\mathscr{E}',\mathscr{F}')=(\mathscr{E}_0^{(\mathtt{s}_2,m,k_2)},\mathscr{F}_0^{(\mathtt{s}_2,m,k_2)}),$$

where $\mathtt{s}_1,\mathtt{s}_2\in\mathscr{S}(I)$ and k_1, k_2 are two Radon measures on I. Now comes our main result.

Theorem 5.6.3 Let $(\mathscr{E}',\mathscr{F}')$ and $(\mathscr{E},\mathscr{F})$ be two local irreducible regular Dirichlet spaces on $L^2(I,m)$. Then $(\mathscr{E}',\mathscr{F}')$ is a regular subspace of $(\mathscr{E},\mathscr{F})$ if and only if

(1) $k_1=k_2$,

(2) $d\mathtt{s}_2$ is absolutely continuous with respect to $d\mathtt{s}_1$ and the density $d\mathtt{s}_2/d\mathtt{s}_1$is either 1 or 0 a.e. $d\mathtt{s}_1$.

Proof. It suffices to prove it for the case that both $(\mathscr{E}', \mathscr{F}')$ and $(\mathscr{E}, \mathscr{F})$ are strongly local. Assume that $\mathscr{F}' \subseteq \mathscr{F}$ and let (X_t, P^x) and (X'_t, P'_x) be the diffusion processes associated with $(\mathscr{E}, \mathscr{F})$ and $(\mathscr{E}', \mathscr{F}')$, respectively. For any $a < c < x_0 < d < b$, define

$$u^{x_0}_{\{c,d\}}(x) := \mathsf{P}'_x(T_{x_0} < T_{\{c,d\}}).$$

We have $u^{x_0}_{\{c,d\}}(x) \in \mathscr{F}' \subseteq \mathscr{F}$, and it shows that $u^{x_0}_{\{c,d\}}(x)$ is absolutely continuous with respect to $\mathtt{s}_1$, while $u^{x_0}_{\{c,d\}}$ is a linear transformation of $\mathtt{s}_2$ on (c, x_0). It follows that $d\mathtt{s}_2$ is absolutely continuous with respect to $d\mathtt{s}_1$ on (c, x_0). Similarly it is also true on (x_0, d). Taking $(c, d) \uparrow (a, b)$, it follows that $d\mathtt{s}_2$ is absolutely continuous with respect to $d\mathtt{s}_1$. Let $f := d\mathtt{s}_2/d\mathtt{s}_1$. Then we have

$$\begin{aligned}
\mathscr{E}'(u,v) &= \int_I \frac{du}{d\mathtt{s}_2}\frac{dv}{d\mathtt{s}_2} d\mathtt{s}_2; \\
\mathscr{E}(u,v) &= \int_I \frac{du}{d\mathtt{s}_1}\frac{dv}{d\mathtt{s}_1} d\mathtt{s}_1 \\
&= \int_I \frac{du}{d\mathtt{s}_2}\frac{dv}{d\mathtt{s}_2} f^2 d\mathtt{s}_1 \\
&= \int_I \frac{du}{d\mathtt{s}_2}\frac{dv}{d\mathtt{s}_2} f d\mathtt{s}_2
\end{aligned}$$

for any $u, v \in \mathscr{F}'$. It follows then that $f d\mathtt{s}_1 = f^2 d\mathtt{s}_1$ and that either $f = 0$ or $f = 1$ a.e. with respect to $d\mathtt{s}_1$. Since $\mathtt{s}_1$ and $\mathtt{s}_2$ are continuous and strictly increasing, f has the property that for any $x, y \in I$ with $x < y$,

$$\int_x^y 1_{\{f=1\}} d\mathtt{s}_1 > 0. \tag{5.6.2}$$

The converse is obvious from the above discussion. □

Let now

$$(\mathscr{E}, \mathscr{F}) = (\mathscr{E}_0^{(\mathtt{s},m,k)}, \mathscr{F}_0^{(\mathtt{s},m,k)})$$

be a local irreducible regular Dirichlet spaces on $L^2(I, m)$. Take a Borel set A having property that for any $x, y \in I$ with $x < y$,

$$\int_x^y 1_{A^c} d\mathtt{s} > 0. \tag{5.6.3}$$

Define $d\mathtt{s}_0 = 1_{A^c} \cdot d\mathtt{s}$. Then $\mathtt{s}_0 \in \mathscr{S}(I)$ and $(\mathscr{E}_0^{(\mathtt{s}_0,m,k)}, \mathscr{F}_0^{(\mathtt{s}_0,m,k)})$ is a regular subspace of $(\mathscr{E}, \mathscr{F})$. It is easy to check that

$$\mathscr{F}_0^{(\mathtt{s}_0,m,k)} = \{u \in \mathscr{F} : \ du/d\mathtt{s} = 0 \text{ a.e. with respect to } d\mathtt{s} \text{ on } A\}.$$

Hence we have a corollary.

Corollary 5.6.1 For any Borel set A satisfying (5.6.3),

$$\mathscr{F}^A = \{u \in \mathscr{F} : \ du/d\mathtt{s} = 0 \text{ a.e. with respect to } d\mathtt{s} \text{ on } A\} \tag{5.6.4}$$

is a regular subspace of $(\mathscr{E}, \mathscr{F})$. Conversely any regular subspace of $(\mathscr{E}, \mathscr{F})$ is induced by such a set.

Finally, we shall give two interesting examples. The first example is a local irreducible and regular Dirichlet space which takes the Dirichlet space $(H^1([0,1]), \frac{1}{2}\mathbf{D})$ of reflected Brownian motion on $[0,1]$ as a proper regular subspace.

Example 1. Let $c(x)$ be the standard Cantor function on $[0,1]$ and let $\mathtt{s}(x) := x + c(x)$. Take m to be the Lebesgue measure on $[0,1]$. Then the Dirichlet space $(H^1([0,1]), \frac{1}{2}\mathbf{D})$, corresponding to Brownian motion on $[0,1]$, is a regular subspace of $(\mathscr{F}^{(\mathtt{s},m)}, \frac{1}{2}\mathscr{E}^{(\mathtt{s},m)})$ by the theorem above and $H^1([0,1])$ is properly contained in $\mathscr{F}^{(\mathtt{s},m)}$.

The second example shows that 1-dim Brownian motion has a non-conservative regular subspace. For this we state a criterion for irreducible one-dimensional diffusions to be conservative (see [36]). Let

$$(\mathscr{E}, \mathscr{F}) = (\mathscr{E}_0^{(\mathtt{s},m,k)}, \mathscr{F}_0^{(\mathtt{s},m,k)})$$

where k is a Radon measure on I and $\mathtt{s} \in \mathscr{S}(I)$, be a local, irreducible and regular Dirichlet space on $L^2(I,m)$ and $X = (X_t, \mathsf{P}^x)$ the associated diffusion. In this case it is either recurrent or transient. We call the left endpoint a of I is

(1) of the first class if a is finite and $a \in I$;

(2) of the second class if $a \notin I$ and $\mathtt{s}(a) = -\infty$;

(3) of the third class if $a \notin I$ and $\mathtt{s}(a) > -\infty$.

We call a is dissipative if a is of the third class and

$$\int_a^c (\mathtt{s}(x) - \mathtt{s}(a))m(dx) < \infty \tag{5.6.5}$$

for some $c \in I$, and hence for all $c \in I$. Obviously, the finiteness (5.6.5) is independent of the choice of the scale function $\mathtt{s}$ and the point c. If a is not dissipative, we call it conservative. The dissipativeness and conservativeness for the right endpoint may be defined similarly. Fix a point $c > a$, define $M(x) := m((x, c))$ for $a < x < c$.

Lemma 5.6.3 The left end-point a is dissipative if and only if a is of the third class and

$$\int_a^c M(x)d\mathtt{s}(x) < \infty. \tag{5.6.6}$$

If a is dissipative, $\lim_{x\downarrow a} M(x)\mathtt{s}(x) = 0$. Similar conclusions hold for the right end-point.

Next, we shall prove sufficient and necessary conditions for one-dimensional diffusion to be transient, recurrent or conservative. We will give a proof using theory of Dirichlet forms.

Theorem 5.6.4 The Dirichlet space $(\mathscr{E}, \mathscr{F})$ (or X) is

(1) recurrent if and only if $k = 0$ and both endpoints are of the first class or the second class;

(2) conservative if and only if $k = 0$ and both endpoints are conservative.

Proof. (1) is easy to be proved. In fact, if $k = 0$, we can construct $(X_t, \mathsf{P}^x), x \in I$ from Brownian motion on $J = \langle \mathtt{s}(a), \mathtt{s}(b) \rangle$ by a time change and a transform of state space, where $\langle , \rangle$ means that the endpoints may be open or closed. When both endpoints are of the first or the second class, the endpoints of J are closed or equal to infinity. But time change and the transform of state space don't change recurrence and transience, it proves (1) when $k = 0$. If $k \neq 0$, for any $u_n \in \mathscr{F} \cap C_c(I)$ with $u_n \uparrow 1$, $\mathscr{E}(u_n, u_n) \geq$

$(u_n, u_n)_k$ and $(u_n, u_n)_k$ is increasing. It shows that $\mathscr{E}(u_n, u_n) \to 0$ doesn't hold.

(2) We have shown that if $k \neq 0$, $(\mathscr{E}, \mathscr{F})$ is transient. Now we shall prove that in this case it is not even conservative. For any sequence $\{u_n\} \subset \mathscr{F}$ with $0 \leq u_n \leq 1$, $u_n \uparrow 1$ m-a.e., take three points a', b', c' with $a < a' < c' < b' < b$, $k((a', b')) > 0$ and

$$\lim_{n\to\infty} u_n(a') = \lim_{n\to\infty} u_n(b') = \lim_{n\to\infty} u_n(c') = 1.$$

Define a function $v \in \mathscr{F}$ as follows:

$$\begin{aligned} v(x) &= \frac{\mathtt{s}(x) - \mathtt{s}(a')}{\mathtt{s}(c') - \mathtt{s}(a')} \text{ if } a' \leq x \leq c', \\ &= \frac{\mathtt{s}(x) - \mathtt{s}(b')}{\mathtt{s}(c') - \mathtt{s}(b')} \text{ if } c' \leq x \leq b' \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} \mathscr{E}(u_n, v) &= \lim_{n\to\infty} (u_n, v)_k + C \int_{a'}^{b'} \frac{du_n}{d\mathtt{s}} \frac{dv}{d\mathtt{s}} d\mathtt{s} \\ &= (1, v)_k + C \lim_{n\to\infty} \left(\frac{u_n(c') - u_n(a')}{\mathtt{s}(c') - \mathtt{s}(a')} + \frac{u_n(c') - u_n(b')}{\mathtt{s}(b') - \mathtt{s}(c')}\right) \\ &= \int_{a'}^{c'} v(x) k(dx) + 0 > 0, \end{aligned}$$

since $v > 0$ on (a', b') and $k((a', b')) > 0$. It shows that $(\mathscr{E}, \mathscr{F})$ is not conservative.

Assumed that a is dissipative and b is an endpoint of the first class. Let $\mathtt{s}$ be a scale function with $\mathtt{s}(a) = 0$. By Lemma 4.1, we have that $\mathtt{s} \in L^1(I, m) \cap \mathscr{F}$. For any function u_n with $0 \leq u_n \leq 1$ and $u_n \uparrow 1$ m-a.e., take a point b' such that $u_n(b') \uparrow 1$ and let $v(x) := \mathtt{s}(x)$ if $a \leq x < b'$, $v(x) = v(b')$ if $b' \leq x < b$,

$$\mathscr{E}(u_n, \mathtt{s}) = \int_a^{b'} \frac{du_n}{d\mathtt{s}} \frac{d\mathtt{s}}{d\mathtt{s}} d\mathtt{s} = u_n(b') \uparrow 1.$$

It proves that $(\mathscr{E}, \mathscr{F})$ is not conservative.

Finally let us prove the converse is true. Assume without loss of generality that a is conservative and b is an endpoint of the first class. For any $c \in (a, b]$, define

$$M^c(x) = m((x, c]), \; M_c(x) = m((c, x]).$$

The conservativeness guarantees that

$$\int_{a'}^{c} M^c(x) d\mathbf{s}(x) \longrightarrow \infty$$

as $a' \downarrow a$ for any c. Take any points c, d with $a < c < d \le b$. Assume at first that m does not charge singleton or M is continuous. Then there exists uniquely $e \in (c, d)$ such that $m(c, e) = m(e, d)$. Define a function

$$w_{c,d}(x) := \begin{cases} M_c(x), & x \in (c, e], \\ M^c(x), & x \in (e, d), \\ 0, & \text{elsewhere.} \end{cases}$$

Obviously $w_{c,d} \ge 0$ is continuous and it may be written as

$$w_{c,d}(x) = \int_a^x \left(1(c, e] - 1(e, d)\right) dm.$$

The conservativeness implies that starting from d we may choose c such that

$$\int_a^b w_{c,d}(x) d\mathbf{s}(x) \ge 1.$$

In general M may not be continuous. But for any $d \in I$, by a delicate analysis, we can still find a point $c \in (a, d)$ and a right continuous non-negative function w on I satisfying

(w1) w is continuous at c and d;

(w2) $w = 0$ on $(a, c]$ and $[d, b]$;

(w3) there exists a simple function w' supported on $[c, d]$ with $|w'| \le 1$ such that $w(x) = \int_a^x w' dm$;

(w4) $\int_a^b w(x)d\mathtt{s}(x) \geq 1$.

Now we start from any point $a_1 \in (a, b]$. There exists $a_2 \in (a, a_1)$ and a function w_{a_2,a_1} satisfying the four conditions above. Hence we have a sequence $\{a_n\}$ which decreases strictly to a such that

$$\int_{a_{n+1}}^{a_n} w_{a_{n+1},a_n} d\mathtt{s} \geq 1$$

for any n.

Define now

$$u_n := \frac{1}{q_n}\int_a^x w_{a_{n+1},a_n} d\mathtt{s},\ x \in (a, b]$$

with

$$q_n = \int_{a_{n+1}}^{a-n} w_{a_{n+1},a_n} d\mathtt{s}.$$

It is not hard to verify that $u_n \in \mathscr{F}$ and $u_n \uparrow 1$ on $(a, b]$. To check the conservativeness of $(\mathscr{E}, \mathscr{F})$, choose any $v \in \mathscr{F} \cap L^1(I, m)$. Then using integration by parts and by the condition (w2) above,

$$\begin{aligned}
\mathscr{E}(u_n, v) &= \frac{1}{q_n}\int_a^b w_{a_{n+1},a_n} dv \\
&= \frac{1}{q_n}\int_a^b v dw_{a_{n+1},a_n} \\
&= \frac{1}{q_n}\int_a^b v \cdot w'_{a_{n+1},a_n} dm,
\end{aligned}$$

where w'_{a_{n+1},a_n} is chosen to satisfy (w3) above. Note that $v \in L^1(I, m)$ and $q_n \geq 1$, $|w'_{a_{n+1},a_n}| \leq 1$. Therefore

$$\sum n \left|\int_a^b v\frac{1}{q_n} w'_{a_{n+1},a_n} dm\right| \leq \int_a^b |v| \sum_n \left|\frac{w'_{a_{n+1},a_n}}{q_n}\right| dm \leq \int_a^b |v| dm < \infty.$$

It follows that $\mathscr{E}(u_n, v) \longrightarrow 0$. That completes the proof. □

We now give an example which illustrates that the Dirichlet space $(\frac{1}{2}\mathbf{D}, H_0^1(\mathbf{R}))$ of Brownian motion on the real line $\mathbf{R}$ has non-conservative regular subspaces, comparing an example in [7] which shows Brownian motion has transient regular subspaces.

Example 2. Define a local irreducible and regular Dirichlet space $(\mathscr{E}_0^{(\mathtt{s},m)}, \mathscr{F}_0^{(\mathtt{s},m)})$ on $L^2(\mathbf{R}, m)$, where m is the usual Lebesgue measure, by giving a scale function

$$\mathtt{s}(x) = \int_0^x 1_G(y)dy, \ x \in \mathbf{R},$$

where

$$G = \bigcup_{r_n \in Q} \left(r_n - \frac{1}{2^{n+1}}, r_n + \frac{1}{2^{n+1}} \right), \tag{5.6.7}$$

where Q is the set of positive rational numbers. We choose an order on Q as follows: if $a, b \in Q$, and $a = \frac{q_1}{p_1}$, $b = \frac{q_2}{p_2}$ take the simplest form, we define

$$a \prec b \Leftrightarrow \text{either } p_1 + q_1 < p_2 + q_2 \text{ or } p_1 + q_1 = p_2 + q_2 \text{ and } q_1 < q_2.$$

Then the order $\prec$ makes Q a sequence $\{r_n\}$ in (5.6.7). Clearly $r_n \leq n$. Thus

$$\int_0^\infty x d\mathtt{s}(x) \leq \sum_n \int_{(r_n - \frac{1}{2^{n+1}}, r_n + \frac{1}{2^{n+1}})} x dx = \sum_n \frac{r_n}{2^n} \leq \sum_n \frac{n}{2^n} < \infty.$$

This shows the right endpoint is dissipative. Therefore the associated process is not conservative.

Chapter 6

Dirichlet forms perturbed by additive functionals

6.1 Introduction

In recent years there were many authors in theory of Dirichlet forms and related fields who studied the so-called Feynman-Kac semigroups, Schrödinger operators and the corresponding bilinear forms. Particularly, the multiplicative functionals in consideration are not necessarily the exponential of classical positive continuous additive functionals or abbreviated as PCAF's. In a serie of papers by Albeverio and Ma ([1], [2] and references therein) they investigated the perturbation of Dirichlet forms by signed smooth measures $\mathscr{E}^\mu = \mathscr{E} + Q_\mu$, where μ is a signed smooth measure and $Q_\mu(f,g) = \mu(f \cdot g)$, and found necessary and sufficient conditions for $\mathscr{E}^\mu$ to be a lower semi-bounded closed quadratic form. In [50] the author studied the killing transformation by general decreasing multiplicative functionals and perturbation of Dirichlet forms by bivariate smooth measures: $\mathscr{E}^\nu = \mathscr{E} + Q_\nu$, where ν is a bivariate smooth measure and $Q_\nu(f,g) = \nu(f \otimes g)$, and proved the generalized Feynman-Kac formula. In [44] the author also studied the additive functionals in the form of $A_t = A_t^\mu + \sum_{s\leq t} F(X_{s-}, X_s)$, where μ is a signed smooth measure, A^μ the difference of two PCAF's associated with μ and F a bounded Borel function vanishing on the diagonal, but his base processes are symmetric stable processes R^d. He found the conditions for the Feynman-Kac semigroup $Q_t f(x) := \mathsf{P}^x(e^{-A_t} f(X_t))$ to be strongly continuous and the bilinear form corresponding to it. Very recently Stollmann and Voigt [46] made a thorough investigation on perturbation by a signed smooth measure, however their approach is rather analytic and even made no assumption of existence of the associated process. In [22] the author studies the generalized Schrödinger equation (attached with a measure) was investigated in context of right Borel Markov processes, but contrary to [46] the approach there are totally probabilistic.

In this chapter we are going to investigate the perturbation of a symmetric Markov process by a general increasing additive functional. More precisely let X be an m-symmetric Markov process on state space E and A an (increasing, symmetric) additive functional of X. Let $(\operatorname{Exp} A)$ be the Stieltjes exponential of A and set $P_t^{-A} f(x) := \mathsf{P}^x[(\operatorname{Exp} A)_t f(X_t)]$ for measurable function f on E and $x \in E$. Then (P_t^{-A}) is a semigroup of kernels which is not Markovian in general. We shall introduce so called additive functionals of extended Kato class as analogous to the notion in [22] and prove that if A belongs to this class, then (P_t^{-A}) may be extended into a strongly continuous semigroup of bounded operators on $L^2(E, m)$. Our approach is very different from the one employed in [22]. We shall also characterize the bilinear form associated with (P_t^{-A}).

I would like to thank P.J. Fitzsimmons and R.K. Getoor for many illuminating discussions and suggestions which, in particular, shape up the right form of the key Lemma 6.2.1.

Notations and Conventions: We use ':=' as a way of definition, which is always read as 'is defined to be'. For a class $\mathscr{F}$ of functions, we denote by $b\mathscr{F}$ (resp. $p\mathscr{F}(= \mathscr{F}^+)$) the set of bounded (resp. nonnegative) functions in $\mathscr{F}$. We won't distinguish 'nonnegative' and 'positive'. When a number $a > 0$ or a function $f > 0$ everywhere, we say they are strictly positive. For a measure μ and a function f, $\mu(f) := \int f d\mu$. We sometimes write L^p or $L^p(m)$ for $L^p(E, m)$ and $(\cdot, \cdot)$ for the inner product in $L^2(m)$. For $f, g \in \mathscr{B}(E)$,

$f \otimes g(x, y) := f(x)g(y)$, $x, y \in E$. Finally we shall use exclusively P^x for both probability measure and expectation.

6.2 Additive functionals of extended Kato class

Throughout this paper $(\mathscr{E}, \mathscr{F})$ is a quasi-regular symmetric Dirichlet form on $L^2(E, m)$, where E is a Lusin space and m a Borel measure on E. Let $\mathscr{S}$ be the set of all smooth measures on E and $\mathscr{S}_0$ the subset of $\mathscr{S}$ consisting of all Borel measures of finite energy integral. Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, X_t, \mathsf{P}^x)$ be a Borel right process on E with transition semigroup (P_t) which is m-symmetric and associated with $(\mathscr{E}, \mathscr{F})$. Let ζ be the life time of X. For a Borel function f on E (write $f \in \mathscr{B}(E)$ sometimes) we set

$$\|f\|_Q = \inf_{\mathrm{Cap}(N)=0} \sup_{x \notin N} |f(x)|, \tag{2.1}$$

where $\mathrm{Cap}(N)$ denotes the 1-capacity of N with respect to $(\mathscr{E}, \mathscr{F})$. When f is quasi-continuous, $\|f\|_Q$ is the same as $\|f\|_\infty$ the usual L^∞-norm. In fact it is clear that $\|f\|_\infty \le \|f\|_Q$. Suppose that $\|f\|_\infty < \|f\|_Q$. Then there exists an m-null set K such that $\sup_{x \notin K} |f(x)| < \|f\|_Q$. We may pick r with $\sup_{x \notin K} |f(x)| < r < \|f\|_Q$ and set $K_1 := \{x \in E : \ |f(x)| > r\}$. Then $K_1 \subset K$ and K_1 is finely open. Thus $\mathrm{Cap}(K_1) = 0$ since K_1 is also an m-null set. We have $\sup_{x \notin K_1} |f(x)| \le r < \|f\|_Q$, which is a contradiction.

A subset N of E is called an exceptional set if $\mathrm{Cap}(N) = 0$. A subset Λ of Ω is called an Ω-equivalent set if there exists an exceptional set N such that $\mathsf{P}^x(\Lambda) = 1$ for all $x \notin N$. We say that A is an additive functional of X if $A = (A_t)_{t \ge 0}$ is a $[0, \infty]$-valued adapted process on Ω and there exists an Ω-equivalent set Λ such that for all $\omega \in \Lambda$, (i) $A_t(\omega) < \infty$ for $t < \zeta(\omega)$; (ii) $t \mapsto A_t(\omega)$ is right continuous; (iii) $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$ for all $t, s \ge 0$. Let $\mathscr{A}$ be the set of all additive functionals of X. Therefore all additive functionals talked in this paper is assumed to be increasing.

It is well-known (see §73 in [37]) that there exists a positive continuous additive functional (abbreviated as PCAF) H of X having bounded 1-potential and a kernel N on $(E, \mathscr{B}(E))$ such that (i) $N(x, \{x\}) = 0$

for all $x \in E$; (ii) for every non-negative Borel function f on $E \times E$, $(\int_0^t Nf(X_s)dH_s)_{t>0}$, where $Nf := \int_E N(\cdot, dy)f(\cdot, y)$, is the dual predictable projection of the random measure

$$\kappa(\omega, dt) := \sum_{s>0} f(X_{s-}(\omega), X_s(\omega))1_{\{X_{s-}(\omega)\neq X_s(\omega)\}}\epsilon_s(dt).$$

We call the pair (N, H) a Lévy system of X. Let $J(dx, dy) := N{*}\rho(dx, dy) := N(x, dy)\rho(dx)$ (noting that the second equality gives a way getting a bivariate measure via a kernel and a measure), where ρ is the Revuz measure of H with respect to m. Then J is the canonical measure of X with respect to m or sometimes called Lévy (also, jumping) measure of $(\mathscr{E}, \mathscr{F})$ and J is symmetric; i.e., $J(dx, dy) = J(dy, dx)$.

Let $\mathscr{B}(E \times E)$ be the set of Borel functions on $E \times E$. The bivariate Revuz measure of $A \in \mathscr{A}$ with respect to m is defined by

$$\nu_A(f) := \uparrow \lim_{t\downarrow 0} \frac{1}{t}\mathsf{P}^m \int_{]0,t]} f(X_{s-}, X_s)dA_s \tag{6.2.1}$$

for $f \in p\mathscr{B}(E \times E)$, it follows from [51] that there exist a smooth measure μ and a nonnegative function F on $E \times E$ vanishing on diagonal such that

$$\nu_A(dx, dy) = \delta * \mu(dx, dy) + F(x, y)J(dx, dy) \tag{6.2.2}$$

where δ denotes the unit kernel $\delta(x, B) := 1_B(x)$, and $A_t = A_t^\mu + \sum_{s\le t} F(X_{s-}, X_s)$ a.e. P^m for any $t > 0$, where A^μ is the PCAF determined by μ. For simplicity we write this as $A = A^{\mu+F}$. Actually μ and F are uniquely determined by A. We say that A is symmetric (with respect to m) if $F \cdot J$ is symmetric as a measure on $E \times E$. Since J is symmetric, we may (and do) choose F as a symmetric function on $E \times E$ with $A = A^{\mu+F}$.

Let L be a right continuous increasing function on $[0, \infty[$ with $L_0 = 0$ which may take infinite value. The unique solution Z of the equation

$$Z_t = 1 + \int_{]0,t]} Z_{s-}dL_s,\ t > 0, \tag{6.2.3}$$

is usually called the Stieltjes exponential function of L, denoted by $(\operatorname{Exp} L)_t$, which coincides with the usual exponential function if L is continuous. The reason we write $(\operatorname{Exp} L)_t$ instead of $\operatorname{Exp} L_t$ is that the Stieltjes exponential is really defined by paths. It is known (see [37]) that if L^c is the continuous part of L, then

$$(6.2.4) \qquad (\operatorname{Exp} L)_t = e^{L_t^c} \prod_{s \le t} (1 + \Delta L_s),$$

where $\Delta L_s = L_s - L_{s-}$. Clearly $\exp(L_t) \ge (\operatorname{Exp} L)_t$ and the equality holds only if L is continuous.

A smooth measure μ is said to belong to the Kato class, which extends the classical notion of Kato class for Brownian motion, if $\lim_{t\downarrow 0} \|\mathsf{P}^\cdot A_t^\mu\|_Q = 0$. However inspired by Getoor [22] and Stollmann-Voigt [46], we may actually go a little further as introducing the so-called additive functionals of extended Kato class, which seems more natural to work with in this context. Given an additive functional A, define

$$(6.2.5) \qquad \begin{aligned} k_t(A) &:= \|\mathsf{P}^\cdot A_t\|_Q; \\ k(A) &:= \inf_{t>0} k_t(A). \end{aligned}$$

Khas'minskii's lemma says that if A is continuous and $\mathsf{P}^x A_t \le a < 1$ for all $x \in E$ and a fixed $t > 0$, then

$$\mathsf{P}^x e^{A_t} \le \frac{1}{1-a}.$$

This is not true when A is not continuous as shown in an example in §6.4. However the following lemma shows that it is true if we replace the usual exponential with the Stieltjes exponential.

Lemma 6.2.1 (a) Let $A \in \mathscr{A}$. If there exist $t > 0$ and $\lambda < 1$ such that for all $s < t$ and $x \in E$, $\mathsf{P}^x(A_t) \le \lambda < 1$, then for $x \in E$,

$$(6.2.6) \qquad \mathsf{P}^x (\operatorname{Exp} A)_t \le \frac{1}{1-\lambda}.$$

(b) Let $A \in \mathscr{A}$. If $k(A) < 1$, there exist positive constants c and β such that for all $t > 0$,

$$(6.2.7) \qquad \|\mathsf{P}^\cdot (\operatorname{Exp} A)_t\|_Q \le c \cdot e^{\beta t}.$$

Proof. (a) From §3 of [5], $\operatorname{Exp} A$ may be developped as

$$(\operatorname{Exp} A)_t = \sum_{n\geq 0} \int_{]0,t]} dA_{t_n} \int_{]0,t_n[} dA_{t_{n-1}} \cdots \int_{]0,t_2[} dA_{t_1}. \tag{6.2.8}$$

Reordering the integrations, we have

$$(\operatorname{Exp} A)_t = \sum_{n\geq 0} \int_{0<t_1<\cdots<t_n\leq t} dA_{t_1} \cdots dA_{t_n}. \tag{6.2.9}$$

Now taking the expectation of the integration and noting that for $\mathsf{P}^x(A_t - A_s|\mathscr{F}_s) = P^{X_s}(A_{t-s}) \leq \lambda$ for $s < t$,

$$\begin{aligned}
&\mathsf{P}^x \int_{0<t_1<\cdots<t_n\leq t} dA_{t_1} \cdots dA_{t_n} \\
&= \mathsf{P}^x \int_{0<t_1<\cdots<t_{n-1}<t} dA_{t_1} \cdots dA_{t_{n-1}} (A_t - A_{t_{n-1}}) \\
&= \mathsf{P}^x \int_{0<t_1<\cdots<t_{n-1}<t} dA_{t_1} \cdots dA_{t_{n-1}} \mathsf{P}^x(A_t - A_{t_{n-1}}|\mathscr{F}_{t_{n-1}}) \\
&\leq \lambda \mathsf{P}^x \int_{0<t_1<\cdots<t_{n-1}\leq t} dA_{t_1} \cdots dA_{t_{n-1}} \\
&\leq \lambda^n.
\end{aligned}$$

Then (6.2.6) follows easily.

(b) Since $k(A) < 1$, we may choose an exceptional set $N \subset E$, $\lambda < 1$ and $T > 0$ such that if starting from any point in $E - N$, the process X never reaches N, and $\mathsf{P}^x A_T \leq \lambda$ for all $x \notin N$. By (a) it holds that $\mathsf{P}^x \operatorname{Exp}(A_T) \leq \frac{1}{1-\lambda}$ for $x \notin N$.

Let $M := (\operatorname{Exp} A)$, which is an increasing multiplicative functional of X. Then for any integer $n \geq 1$ we have

$$\begin{aligned}
\mathsf{P}^x M_{nT} &= \mathsf{P}^x\{M_T \circ \theta_{(n-1)T} M_{(n-1)T}\} \\
&= \mathsf{P}^x\{M_{(n-1)T} E^{X_{(n-1)T}} M_T\} \\
&\leq \frac{1}{1-\lambda} \mathsf{P}^x M_{(n-1)T} \leq (\frac{1}{1-\lambda})^n.
\end{aligned}$$

For any $t > 0$, take n such that $(n-1)T < t \leq nT$. It follows that

$$\mathsf{P}^x M_t \leq \mathsf{P}^x M_{nT} \leq (\frac{1}{1-\lambda})^n \leq (\frac{1}{1-\lambda})^{\frac{t}{T}+1}.$$

Now (6.2.7) holds for $c = \frac{1}{1-\lambda}$ and $\beta = \frac{1}{T}\log\frac{1}{1-\lambda}$. □

Remark. The argument in the proof actually proves a little more. If L is a right continuous adapted increasing process on Ω with $L_0 = 0$ and, for $x \in E$ and $s < t$, $\mathsf{P}^x(L_t - L_s|\mathscr{F}_s) \le \lambda < 1$, then

$$\mathsf{P}^x(\operatorname{Exp} L)_t \le \frac{1}{1-\lambda}.$$

The readers may compare it to a result of Dellacherie and Meyer: if $\mathsf{P}^x(L_t - L_{s-}|\mathscr{F}_s) \le \lambda < 1$, then

$$\mathsf{P}^x e^{L_t} \le \frac{1}{1-\lambda},$$

from which the Khasminskii's lemma follows easily. We may also feel the difference between two exponentials.

Let $\mathscr{P}$ be the set of all symmetric additive functionals of X and define

$$\text{(6.2.10)} \qquad \begin{aligned} \mathscr{P}_0 &:= \{A \in \mathscr{P} :\ \rho_A \text{ is smooth}\} \\ \mathscr{P}_K &:= \{A \in \mathscr{P} :\ k(A) < 1\}. \end{aligned}$$

The element in $\mathscr{P}_K$ is called an additive functionals of the extended Kato class. A smooth measure μ (resp. $F \in p\mathscr{B}(E \times E)$) with $A^\mu \in \mathscr{P}_K$ (resp. $A^F \in \mathscr{P}_K$) is said to belong to the extended Kato class. Let A^* be the dual predictable projection (compensator) of A. Clearly if $A \in \mathscr{P}_0$, A^* is a PCAF of X and in that case $A^* = A^\mu + \int NFdH$ as $A = A^{\mu+F}$. It is easily to see that $\mathsf{P}^x A_t = \mathsf{P}^x A_t^*$ for all $t > 0$ and $x \in E$, $\rho_A = \rho_{A^*}$ and if $A \in \mathscr{P}_K$, A^* is a PCAF of the extended Kato class.

Suppose that $A \in \mathscr{P}$. We define the A-perturbation semigroup of (P_t) (or, X) as

$$\text{(6.2.11)} \qquad P_t^{-A} f(x) := \mathsf{P}^x((\operatorname{Exp} A)_t f(X_t)),\ f \in p\mathscr{B}(E),\ x \in E,$$

(we use $-A$ to be consistent with the standard notation P_t^q) and the A-perturbation bilinear form $(\mathscr{E}^{-A}, \mathscr{F}^{-A})$ (or, (X, m)) as

$$\text{(6.2.12)} \qquad \begin{aligned} \mathscr{F}^{-A} &:= \mathscr{F} \cap L^2(\rho_A), \\ \mathscr{E}^{-A}(u,u) &:= \mathscr{E}(u,u) - \nu_A(u \otimes u),\ u \in \mathscr{F}^{-A}, \end{aligned}$$

where ρ_A denotes the Revuz measure of A, which equals the marginal measure of ν_A. It follows from the additivity of A that (P_t^{-A}) is a semigroup of kernels on $(E, \mathscr{B}(E))$ and from the Hölder's inequality that $(\mathscr{E}^{-A}, \mathscr{F}^{-A})$ is a well-defined bilinear form on $L^2(m)$.

Lemma 6.2.2 Let A be an additive functional of X. If A is symmetric, then (P_t^{-A}) is m-symmetric; i.e., for all $f, g \in p\mathscr{B}(E)$,

$$(P_t^{-A} f, g) = (f, P_t^{-A} g).$$

Proof. Recall the reversibility of X under P^m. Let (γ_t) be the reversal operators on Ω; namely, for any $\omega \in \Omega$ and $t < \zeta(\omega)$

$$X_s(\gamma_t \omega) := \begin{cases} X_{(t-s)-}(\omega), & s \leq t; \\ \Delta, & s > t, \end{cases}$$

where Δ is the trap point of X. Since X is m-symmetric, it is reversable under P^m; more precisely, for any $t > 0$ and a nonnegative $\mathscr{F}_t$-measurable random variable G,

$$\mathsf{P}^m(G;\ t < \zeta) = \mathsf{P}^m(G \circ \gamma_t;\ t < \zeta). \tag{6.2.13}$$

Since F is symmetric, it is easy to check that $(\operatorname{Exp} A)_t \circ \gamma_t = (\operatorname{Exp} A)_t$ and hence we have

$$\begin{aligned} \mathsf{P}^m((\operatorname{Exp} A)_t f(X_t) g(X_0)) &= \mathsf{P}^m((\operatorname{Exp} A)_t f(X_t) g(X_0)) \circ \gamma_t) \\ &= \mathsf{P}^m((\operatorname{Exp} A)_t f(X_0) g(X_t)). \end{aligned}$$

That completes the proof. □

We are now going to show that if $A \in \mathscr{P}_K$, the perturbation semigroup is actually a strongly continuous semigroup of bounded operators on $L^2(E, m)$. When A is continuous, Getoor [22] proved a much more general and stronger result for perturbation semigroup. Unfortunately his argument does not apply in our situation because the Stieltjes exponential behaves very differently from the usual one in some way. Thus we shall use a rather different approach.

Assume that $A \in \mathscr{P}_0$. Then $A - A^*$ is a local martingale and its Doleans-Dade exponential may be written as

$$M := \mathrm{Exp}(A - A^*) = \{e^{-A_t^*}(\mathrm{Exp}\, A)_t\}_t,$$

which is a supermartingale multiplicative functional of X. Let Y be the subprocess of X transformed by M, which is m-symmetric, and (P_t^Y) the corresponding transition semigroup, which is also a strongly continuous contraction semigroup of bounded operators on $L^2(E, m)$. Let $(\mathscr{E}^Y, \mathscr{F}^Y)$ be the Dirichlet form on $L^2(E, m)$ associated with Y and set

$$\mathscr{F}' := \mathscr{F} \cap \{u \in \mathscr{F} : \int (u(y) - u(x))^2 \nu_A(dx, dy) < \infty\}, \tag{6.2.14}$$

$$\mathscr{E}'(u, u) := \mathscr{E}(u, u) + \frac{1}{2} \int (u(y) - u(x))^2 \nu_A(dx, dy), \ u \in \mathscr{F}'.$$

Lemma 6.2.3 Suppose $A \in \mathscr{P}_0$. Then

(a) $\mathscr{F} \cap L^2(\rho_A)$ is densely contained in $\mathscr{F}^Y$, $\mathscr{F} \cap L^2(\rho_A) = \mathscr{F}^Y \cap L^2(\rho_A)$ and for $u \in \mathscr{F} \cap L^2(\rho_A)$, $\mathscr{E}^Y(u, u) = \mathscr{E}'(u, u)$;

(b) $\mathscr{F}^Y \subset \mathscr{F}'$ and for $u \in \mathscr{F}^Y$, $\mathscr{E}'(u, u) \leq \mathscr{E}^Y(u, u)$.

Proof. (a) Let

$$M_t^+ := (\mathrm{Exp}\, A)_t, \quad M_t^- := \exp(-A_t^*),$$

and Z be the subprocess of X transformed by M^- with the associated Dirichlet form $(\mathscr{E}^Z, \mathscr{F}^Z)$ on $L^2(E, m)$, which is given exactly by

$$\mathscr{F}^Z = \mathscr{F} \cap L^2(\rho_{A^*}); \tag{6.2.15}$$

$$\mathscr{E}^Z(u, u) = \mathscr{E}(u, u) + \rho_{A^*}(u^2), \ u \in \mathscr{F}^Z.$$

Clearly Z coincides with the subprocess of Y transformed by

$$\frac{1}{M^+} = e^{-A_t^\mu}\{\prod_{s \leq t}(1 - \frac{F(X_{s-}, X_s)}{1 + F(X_{s-}, X_s)})\}_t.$$

It is known from [29] that the $((1 + F)N, H)$, where $((1 + F)N)(x, dy) = (1 + F(x, y))N(x, dy)$, is a Lévy system of Y. Hence the jumping measure of

Y equals $(1+F)J$ and the bivariate Revuz measure of $\frac{1}{M^+}$ computed with respect to (Y, m) equals ν_A. It follows from [51] that

$$\mathscr{F}^Z = \mathscr{F}^Y \cap L^2(\rho_A);$$

(6.2.16)

$$\begin{aligned}\mathscr{E}^Z(u,u) &= \mathscr{E}^Y(u,u) + \nu_A(u \otimes u)\\ &= \mathscr{E}^Y(u,u) - \frac{1}{2}\int (u(y)-u(x))^2 \nu_A(dx,dy) + \rho_A(u^2),\ u \in \mathscr{F}^Z.\end{aligned}$$

Combining (6.2.15) and (6.2.16), (a) follows.

(b) Assume that $u \in \mathscr{F}^Y$. We may choose a sequence $\{u_n\} \subset \mathscr{F} \cap L^2(\rho_A)$ such that $u_n \longrightarrow u$ in $\mathscr{E}_q^Y$-norm. Then $\{u_n\}$ is an $\mathscr{E}^Y$-Cauchy sequence and by the result above it is also an $\mathscr{E}$-Cauchy sequence. Therefore $u \in \mathscr{F}$ and $u_n \longrightarrow u$ in $\mathscr{E}_q$-norm and quasi-everywhere (at least for a subsequence). Invoking the Fatou's lemma we have

$$\mathscr{E}'(u,u) \le \lim_n \mathscr{E}'(u_n,u_n) = \lim_n \mathscr{E}^Y(u_n,u_n) = \mathscr{E}^Y(u,u) < \infty.$$

Therefore (b) follows. □

A bilinear form $(b, D(b))$ on $L^2(m)$ is lower semi-bounded if there exists $q > 0$ such that $b(u,u) + q(u,u) \ge 0$ for all $u \in D(b)$. Theorem 4.1 of [2] says that if A is a PCAF, then the A-perturbation semigroup of X is a strongly continuous semigroup on $L^2(m)$ if and only if the A-perturbation bilinear form of X is lower semibounded. The part (a) of the following result generalizes this theorem slightly.

Theorem 6.2.1 Suppose that $A \in \mathscr{P}_0$.

(a) The A-perturbation semigroup of (P_t) is a strongly continuous semigroup on $L^2(m)$ if and only if the A-perturbation bilinear form of $(\mathscr{E}, \mathscr{F})$ is lower semi-bounded.

(b) If the A^*-perturbation semigroup of (P_t) is a strongly continuous semigroup of bounded operators on $L^2(E, m)$, so is the A-perturbation semigroup of (P_t).

Proof. (a) It is obvious that the A-perturbation semigroup of (P_t) is exactly the same as the A^*-perturbation semigroup of (P_t^Y). We denote by $(\mathscr{E}^*, \mathscr{F}^*)$ the A^*-perturbation bilinear form of $(\mathscr{E}^Y, \mathscr{F}^Y)$. By the definition and Lemma 6.2.3 $\mathscr{F}^* = \mathscr{F}^Y \cap L^2(\rho_{A^*}) = \mathscr{F} \cap L^2(\rho_A) = \mathscr{F}^{-A}$ and for $u \in \mathscr{F}^*$,

$$\mathscr{E}^*(u,u) = \mathscr{E}^Y(u,u) - \rho_{A^*}(u^2) = \mathscr{E}(u,u) - \nu_A(u \otimes u) = \mathscr{E}^{-A}(u,u)$$

since $\rho_A = \rho_{A^*}$. It means that the A-perturbation bilinear form of $(\mathscr{E}, \mathscr{F})$ is exactly the same as the A^*-perturbation bilinear form of $(\mathscr{E}^Y, \mathscr{F}^Y)$. Now (a) follows from Theorem 4.1 of [2] applying to A^* and Y.

(b) Given the condition, we know that the A^*-perturbation bilinear form of $(\mathscr{E}, \mathscr{F})$ is lower semi-bounded, that is, there exists $q > 0$ such that for $u \in \mathscr{F}^{-A^*}$,

$$\mathscr{E}_q(u,u) - \rho_{A^*}(u^2) \geq 0.$$

By Lemma 6.2.3, $\mathscr{F}^{-A} = \mathscr{F} \cap L^2(\rho_A) \subset \mathscr{F}^Y$ and for $u \in \mathscr{F}^{-A}$,

$$\rho_{A^*}(u^2) \leq \mathscr{E}_q(u,u) \leq \mathscr{E}_q^Y(u,u);$$

i.e., $\mathscr{E}_q(u,u) - \nu_A(u \otimes u) \geq 0$. It means that the A-perturbation bilinear form of $(\mathscr{E}, \mathscr{F})$ is lower semi-bounded and therefore by (a) it follows that the A-perturbation semigroup of (P_t) is strongly continuous on $L^2(m)$. □

Now comes our main theorem of this section.

Theorem 6.2.2 Suppose that $A \in \mathscr{P}_K$. Then the A-perturbation semigroup of (P_t) is a strongly continuous semigroup of symmetric bounded operators on $L^2(E, m)$.

Proof. That $A \in \mathscr{P}_K$ implies that $A^* \in \mathscr{P}_K$. By Theorem 4.15 of [22], the A^*-perturbation semigroup of (P_t) is a strongly continuous semigroup of bounded operators on $L^2(E, m)$. Hence the conclusion follows from Theorem 6.2.1(b). □

6.3 Perturbation of bilinear forms

In this section we are going to further characterize the relationship between the A-perturbation semigroup and A-perturbation bilinear form of X. First we introduce the resolvent corresponding the perturbation semigroup.

We know that if $A \in \mathscr{A}$, then $\{\frac{1}{(\operatorname{Exp} A)_t}\}_{t>0}$, is a decreasing multiplicative functional of X, which we denote by $(\operatorname{Exp} A)^-$. It is easy to see that $(\operatorname{Exp} A)^-$ does not vanish before ζ and it is the unique solution of the equation

$$Z_t = 1 - \int_0^t Z_s dA_s. \tag{6.3.1}$$

Let $A^1, \cdots A^a, B^1, \cdots B^b, K^1, \cdots K^k, L^1, \cdots L^l \in \mathscr{A}$ and introduce notations as follows.

$$\mathscr{L} := [A^1, \cdots A^a, -B^1, \cdots - B^b, K^1_-, \cdots K^k_-, -L^1_-, \cdots - L^l_-];$$
$$(\operatorname{Exp} \mathscr{L})_t := \prod_{1\le i\le a} (\operatorname{Exp} A^i)_t \prod_{1\le i\le b} (\operatorname{Exp} B^i)^-_t \prod_{1\le i\le k} (\operatorname{Exp} K^i)_{t-} \prod_{1\le i\le l} (\operatorname{Exp} L^i)^-_{t-}.$$

Clearly $\operatorname{Exp} \mathscr{L}$ is still a multiplicative functional of X which does not vanish before ζ. Note that the order in $\mathscr{L}$ is irrelevant and if some elements in $\mathscr{L}$ vanish, they can be simply removed. Then we define

$$\begin{aligned} P_t^{-\mathscr{L}} f(x) &:= \mathsf{P}^x((\operatorname{Exp} \mathscr{L})_t f(X_t)), \\ U_L^{q-\mathscr{L}} f(x) &:= \mathsf{P}^x \int_{]0,\infty[} e^{-qt} (\operatorname{Exp} \mathscr{L})_t f(X_t) dL_t, \end{aligned} \tag{6.3.2}$$

where $f \in p\mathscr{B}(E)$, $x \in E$, $q \ge 0$ and $L \in \mathscr{A}$. Obviuously P_t^{-A} defined in §6.2 coincides with $P_t^{-[A]}$. Thus we also write U_L^{q-A} (resp. $U_L^{q-A_-}$) for $U_L^{q-[A]}$ (resp. $U_L^{q-[A_-]}$). The following lemma gives a few formulas similar to the resolvent equation.

Lemma 6.3.1 Let $A^1, A^2, B^1, B^2, L \in \mathscr{A}$, $q \ge 0$ and $f \in p\mathscr{B}(E)$.

(a) If $U^{q-[A^1,-A^2]} f(x) < \infty$, then

$$\begin{aligned} &U_L^{q-[A^1,-A^2]} f(x) + U_{B^1}^{q-[A^1,-A^2,B^1_-,-B^2_-]} U_L^{q-[A^1,-A^2]} f(x) \\ &= U_L^{q-[A^1,-A^2,B^1_-,-B^2_-]} f(x) + U_{B^2}^{q-[A^1,-A^2,B^1,-B^2]} U_L^{q-[A^1,-A^2]} f(x). \end{aligned}$$

(b) If B^1, B^2 are continuous and $U_L^{q-[A^1,-A^2]} f(x) < \infty$, then

$$\begin{aligned}&U_L^{q-[A^1,-A^2]} f(x) + U_{B^1}^{q-[A^1,-A^2]} U_L^{q-[A^1,-A^2,B^1,-B^2]} f(x) \\ &= U_L^{q-[A^1,-A^2,B^1,-B^2]} f(x) + U_{B^2}^{q-[A^1,-A^2]} U_L^{q-[A^1,-A^2,B^1,-B^2]} f(x).\end{aligned}$$

(c) If $A \in \mathscr{A}$ and $U_L^q f(x) < \infty$, then

$$U_L^{q-A_-} f(x) = U_L^q f(x) + U_A^q U_L^{q-A_-} f(x).$$

Proof. (a) By (6.2.3), (6.3.1) and using the Markovian property,

$$\begin{aligned}&U_{B^1}^{q-[A^1,-A^2,B^1_-,-B^2_-]} U_L^{q-[A^1,-A^2]} f(x) \\ &= \mathsf{P}^x \int_{]0,\infty[} e^{-qt} (\mathrm{Exp}[A^1,-A^2])_t (\mathrm{Exp}\, B^2)^-_{t-} d(\mathrm{Exp}\, B^1)_t \\ &\qquad \cdot \Big(\int_{]0,\infty[} e^{-qs} (\mathrm{Exp}[A^1,-A^2])_s f(X_s) dL_s\Big)\circ\theta_t \\ &= \mathsf{P}^x \int_{]0,\infty[} (\mathrm{Exp}\, B^2)^-_{t-} \int_{]t,\infty[} e^{-qs} (\mathrm{Exp}[A^1,-A^2])_s f(X_s) dL_s d(\mathrm{Exp}\, B^1)_t \\ &= -U_L^{q-[A^1,-A^2]} f(x) \\ &\qquad + \mathsf{P}^x \int_{]0,\infty[} (\mathrm{Exp}[B^1_-,-B^2_-])_t d\Big(-\int_{]t,\infty[} e^{-qs} (\mathrm{Exp}[A^1,-A^2])_s f(X_s) dL_s\Big) \\ &\qquad - \mathsf{P}^x \int_{]0,\infty[} (\mathrm{Exp}\, B^1)_t \int_{]t,\infty[} e^{-qs} (\mathrm{Exp}[A^1,-A^2])_s f(X_s) dL_s d(\mathrm{Exp}\, B^2)^-_t \\ &= -U_L^{q-[A^1,-A^2]} f(x) + U_L^{q-[A^1,-A^2,B^1_-,-B^2_-]} f(x) \\ &\qquad + U_{B^2}^{q-[A^1,-A^2,B^1,-B^2]} U_L^{q-[A^1,-A^2]} f(x).\end{aligned}$$

The proof of (b) is similar.

(c) By a direct computation, we have

$$\begin{aligned}U_A^q U_L^{q-A_-} f(x) &= \mathsf{P}^x \int_{]0,\infty[} e^{-qt} dA_t \Big(\int_{]0,\infty[} e^{-qs+A_{s-}} f(X_s) dL_s\Big)\circ\theta_t \\ &= -\mathsf{P}^x \int_{]0,\infty[} d(\mathrm{Exp}\, A)^-_t \int_{]t,\infty[} e^{-qs} (\mathrm{Exp}\, A)_{s-} f(X_s) dL_s \\ &= U_L^{q-A_-} f(x) - \mathsf{P}^x \int_{]0,\infty[} (\mathrm{Exp}\, A)^-_{t-} e^{-qt} (\mathrm{Exp}\, A)_{t-} f(X_t) dL_t\end{aligned}$$

$$= U_L^{q-A-} f(x) - U_L^q f(x).$$

That completes the proof. $\square$

Now we assume that $A \in \mathscr{P}_K$ and define $\beta(A)$ to be the minimum β such that

$$\|\mathsf{P}^{\cdot}(\operatorname{Exp} A)_t\|_Q \leq c \cdot e^{\beta t}$$

holds for a constant c and all $t > 0$. Clearly $\beta(A) < \infty$.

Theorem 6.3.1 Let $A \in \mathscr{P}_K$ and $q > \beta(A)$.

(a) For all $f \in L^2(m)$, $U^{q-A} f \in \mathscr{F}^{-A}$.

(b) For all $f \in L^2(m)$ and $u \in \mathscr{F}$,

$$(f, u) = \mathscr{E}_q(U^{q-A} f, u) - \nu_A(U^{q-A} f \otimes u).$$

Hence for $u \in \mathscr{F}^{-A}$, $(f, u) = \mathscr{E}_q^{-A}(U^{q-A} f, u)$.

(c) $(\mathscr{E}^{-A}, \mathscr{F}^{-A})$ is a closable lower semibounded bilinear form on $L^2(m)$.

(d) $\mathscr{F}^{-A} = \mathscr{F}$.

(e) If $2k(A) < 1$, then $(\mathscr{E}^{-A}, \mathscr{F}^{-A})$ is closed.

Before proving this theorem, we will present a few lemmas first. We should also mention that many ideas and approaches come directly from [2].

Lemma 6.3.2 Let $A \in \mathscr{P}_K$ and $q > \beta(A)$. Then

$$\|U^{q-A} 1\|_Q + \|U_A^{q-A-} 1\|_Q + \|U_A^q 1\|_Q < \infty. \tag{6.3.3}$$

Proof. It is obvious that since $q > \beta(A)$, $a := \|\mathsf{P}^x \int_0^\infty e^{-qt} (\operatorname{Exp} A)_t dt\|_Q < \infty$. Now it is easily seen that $\|U^{q-A} 1\|_Q \leq a$ and

$$\begin{aligned} U_A^{q-A-} 1 &\leq \mathsf{P}^{\cdot} \int_0^\infty e^{-qt} (\operatorname{Exp} A)_{t-} dA_t \\ &= \mathsf{P}^{\cdot} \int_0^\infty e^{-qt} d(\operatorname{Exp} A)_t = -1 + q \mathsf{P}^{\cdot} \int_0^\infty e^{-qt} (\operatorname{Exp} A)_t dt. \end{aligned}$$

Hence $\|U_A^{q-A-} 1\|_Q \leq qa - 1 < \infty$. Finally $U_A^q 1 \leq U_A^{q-A-} 1$. That completes the proof. $\square$

Lemma 6.3.3 Let $A \in \mathscr{P}_K$ and $q > \beta(A)$.

(a) If $g \in L^2(m + \rho_A)$ and $U_A^q g \in L^2(\rho_A)$, then $U_A^q g \in \mathscr{F}$.

(b) For any $\alpha > 0$, $U^\alpha(L^2(m)) \subset L^2(\rho_A)$.

(c) If $f \in L^2(m)$ and $U^{q-A} f \in L^2(\rho_A)$, then $U^{q-A} f \in \mathscr{F}$.

Proof. (a) Taking the approximating form $\mathscr{E}_q^{(p)}$ of $\mathscr{E}_q$, we have for $g \geq 0$,

$$\begin{aligned}\mathscr{E}_q^{(p)}(U_A^q g, U_A^q g) &= p(U_A^q g, U_A^q g - pU^{q+p} U_A^q g) \\ &= p(U_A^q g, U_A^{q+p} g) \\ &= \nu_A(pU^{q+p} U_A^q g \otimes g).\end{aligned}$$

The last equality follows from the Revuz formula (see, e.g., [23]): for any $u, v \in p\mathscr{B}(E)$,

$$(u, U_A^\alpha v) = \nu_A(U^\alpha u \otimes v). \tag{6.3.4}$$

$U_A^q g$ is q-excessive,

$$\sup_p \mathscr{E}_q^{(p)}(U_A^q g, U_A^q) = \nu_A(U_A^q g \otimes g) < \infty,$$

by the conditions. Hence $U_A^q g \in \mathscr{F}$.

(b) By the Revuz formula (6.3.4) again, for $f \in L^2(m)$,

$$\rho_A[(U^\alpha f)^2] = \nu_A((U^\alpha f)^2 \otimes 1) \leq \frac{1}{\alpha} \nu_A(U^\alpha f^2 \otimes 1) \leq (f^2, U_A^\alpha 1).$$

Thus (b) follows from (6.3.3).

(c) By Lemma 6.3.1(c) it also suffices to show that $U_A^q U^{q-A} f \in \mathscr{F}$ for $f \geq 0$. We know by (b) and Lemma 6.3.1(c) that $U_A^q U^{q-A} f = U^{q-A} f - U^q f \in L^2(\rho_A)$. Hence it follows from (a) that $U_A^q U^{q-A} f \in \mathscr{F}$. □

Remark. Suppose that ξ is a smooth measure of extended Kato class. We may easily see from (b) that $U^\alpha(L^2(m)) \subset L^2(\xi)$ for $\alpha > 0$.

The key to prove that U^{q-A} carries $L^2(m)$ into $\mathscr{F}$ is to prove that it carries $L^2(m)$ into $L^2(\rho_A)$. We need a generalized Revuz formula. Let

$\mathscr{L}^q$ be the q-energy functional of X which is m-symmetric. We list two properties of $\mathscr{L}^q$ which may be checked easily by using the properties of energy functional (see [16]).

L-1. $\mathscr{L}^q$ is m-symmetric in the sense that $\mathscr{L}^q(h_1 m, h_2) = \mathscr{L}^q(h_2 m, h_1)$ for q-excessive functions h_1 and h_2.

L-2. If $A \in \mathscr{A}$, $f \in p\mathscr{B}(E)$ and h is q-excessive, then $\mathscr{L}^q(hm, U_A^q f) = \nu_A(h \otimes f)$.

The Revuz formula (6.3.4) follows easily from L-2.

Lemma 6.3.4 Let $A \in \mathscr{P}_K$ and $q > \beta(A)$. Suppose that $A^1, A^2 \in \mathscr{P}_0$. Then we have a so-called generalized Revuz formula

(6.3.5) $$\nu_{A^1}(U_{A^2}^{q-A-} f_2 \otimes f_1) = \nu_{A^2}(U_{A^1}^{q-A-} f_1 \otimes f_2),$$

for $f_1, f_2 \in p\mathscr{B}(E)$.

Proof. It follows from Lemma 6.3.1(c) that $U_{A^i}^{q-A-} f_i$, $i = 1, 2$ are q-excessive for X. Hence by (L-2) we have

$$\begin{aligned}
&\mathscr{L}^q((U_{A^1}^{q-A-} f_1) \cdot m, U_{A^2}^{q-A-} f_2) \\
&= \mathscr{L}^q((U_{A^1}^{q-A-} f_1) \cdot m, U_{A^2}^q f_2) + \mathscr{L}^q((U_{A^1}^{q-A-} f_1) \cdot m, U_A^q U_{A^2}^{q-A-} f_2) \\
&= \nu_{A^2}(U_{A^1}^{q-A-} f_1 \otimes f_2) + \nu_A(U_{A^1}^{q-A-} f_1 \otimes U_{A^2}^{q-A-} f_2).
\end{aligned}$$

Switching A^1 and A^2 respectively, we also have

$$\begin{aligned}
&\mathscr{L}^q((U_{A^2}^{q-A-} f_2) \cdot m, U_{A^1}^{q-A-} f_1) \\
&= \nu_{A^1}(U_{A^2}^{q-A-} f_2 \otimes f_1) + \nu_A(U_{A^2}^{q-A-} f_2 \otimes U_{A^1}^{q-A-} f_1).
\end{aligned}$$

By symmetry (L-1), we see that (6.3.5) holds as soon as

(6.3.6) $$\nu_A(U_{A^1}^{q-A-} f_1 \otimes U_{A^2}^{q-A-} f_2) < \infty.$$

We first assume $k(A^1) = k(A^2) = 0$. Then $L := A + A^1 \in \mathscr{P}_K$. Let $s > \beta(L)$. Then by Lemma 6.3.4

$$\|U_{A^1}^{s-A-} 1\|_Q \le \|U_L^{s-L-} 1\|_Q < \infty.$$

But mimicking the proof of Lemma 3.1(b) and taking a special case, we have

$$U_{A^1}^{q-A-}1 = U_{A^1}^{s-A-}1 + (s-q)U^{q-A}U_{A^1}^{s-A-}1.$$

which is similar to the resolvent equation. Hence $\|U_{A^1}^{q-A-}1\|_Q < \infty$ and similarly $\|U_{A^2}^{q-A-}1\|_Q < \infty$.

Since $A \in \mathscr{P}_K$, A^* is a PCAF and we may choose, for the smooth measure ρ_{A^*}, an increasing sequence $\{E_n\}$ of subsets of E such that (i) for any $n \geq 1$, $1_{E_n}\rho_{A^*}$ is a finite measure of Kato class; (ii) $\rho_{A^*}(E - \cup_{n=1}^{\infty} E_n) = 0$; (iii) $\lim_n \mathrm{Cap}(K - E_n) = 0$ for any compact set K. Set $B_n := E_n \times E_n$. Then $1_{B_n}\nu_A(E \times E) \leq \rho_{A^*}(E_n) < \infty$ and $1_{B^n}\nu_A \uparrow \nu_A$. Define

$$H_t^n := \int_0^t 1_{B_n}(X_{s-}, X_s)dA_s,\ t > 0.$$

Then $H^n \in \mathscr{P}_K$ and $\nu_{H^n} = 1_{B^n}\nu_A$ for each $n \geq 1$. Thus (6.3.6), and then (6.3.5), holds for bounded f_1, f_2 by replacing A with H^n. Now the monotone convergence theorem (MCT) implies that (6.3.5) holds for all $f_1, f_2 \in p\mathscr{B}$ with B^n in place. Let n tend to infinity. Applying MCT again, (6.3.5) follows under our assumption $k(A^1) = k(A^2) = 0$. To remove this assumption, it suffices to use the arguments above for A^1, A^2 and MCT again. □

Remark. One may prove a slightly more general formula

$$\nu_{A^1}((U_{A^2}^{q-A-} f_2 \otimes 1)\cdot f_1) = \nu_{A^2}((U_{A^1}^{q-A-} f_1 \otimes 1)\cdot f_2),\ f_1, f_2 \in p\mathscr{B}(E \times E) \quad (3.7)$$

where

$$U_{A^1}^{q-A-} f_1 := \mathsf{P}^{\cdot} \int_0^\infty e^{-qt+A_{t-}} f_1(X_{t-}, X_t)dA_t^1,$$

and similar for the other.

Proof of Theorem 6.3.1. (a) We need only to show that for $f \in L^2(m)$, $U^{q-A}f \in L^2(\rho_A)$. Let c be the constant of the lefthand side in (6.3.3). We find by (6.3.5)

$$\begin{aligned}\rho_A(|U^{q-A}f|^2) &\leq c\rho_A(U^{q-A}|f|^2)\\ &= c\nu_A(U^{q-A}|f|^2 \otimes 1)\end{aligned}$$

$$= c\nu_m(U_A^{q-A-}1 \otimes |f|^2)$$
$$\leq c^2 m(|f|^2) < \infty,$$

where ν_m denotes the bivariate Revuz measure of dt which equals $\delta * m$.

(b) Without loss of generality we assume $f, u \geq 0$. First by Lemma 6.3.1(c), we have

$$(f, u) = \mathscr{E}_q(U^q f, u) = \mathscr{E}_q(U^{q-A} f, u) - \mathscr{E}_q(U_A^q U^{q-A} f, u).$$

Employing the approximating form, resolvent equation and Revuz formula,

$$\mathscr{E}_q(U_A^q U^{q-A} f, u) = \lim_p p(U_A^{q+p} U^{q-A} f, u)$$
$$= \lim_p \nu_A(U^{q-A} f \otimes pU^{q+p} u).$$

A similar argument as in the proof of (a) shows that the measure $\nu_A(U^{q-A} f \otimes \cdot)$ is finite if f is L^1-integrable. It is clear that $pU^{q+p}u \longrightarrow u$ q.e. as $p \to \infty$. Let $u_n := u \wedge n$ and $f_n := 1_{E_n}(f \wedge n)$, where $\{E_n\}$ is a sequence chosen for m as in the proof of Lemma 6.3.3. Then $m(f_n) < \infty$ and by the dominated convergence theorem, we have

$$\lim_p \nu_A(U^{q-A} f_n \otimes pU^{q+p} u_l) = \nu_A(U^{q-A} f_n \otimes u_l).$$

Hence

$$(f_n, u_l) + \nu_A(U^{q-A} f_n \otimes u_l) = \mathscr{E}_q(U^{q-A} f_n, u_l). \tag{6.3.7}$$

First $u_l \uparrow u$ q.e. and in $\mathscr{E}_q$-norm. Thus we may erase l in (6.3.7) by MCT. Now $f_n \uparrow f$ a.e.-m, also in L^2, and then $U^{q-A} f_n \uparrow U^{q-A} f$ q.e., also in L^2. A little more computation shows that

$$\mathscr{E}_q(U^{q-A} f_n, U^{q-A} f_n) \leq \mathscr{E}_q(U^{q-A} f, U^{q-A} f).$$

Hence $U^{q-A} f_n \longrightarrow U^{q-A} f$ weakly in $(\mathscr{E}, \mathscr{F})$ and (b) follows.

(c) Let $(\bar{H}, D(\bar{H}))$ be the generator of (P_t^{-A}). Since (P_t^{-A}) is strongly continuous and symmetric, $(q - \bar{H}, D(\bar{H}))$ is a positive definite self adjoint operator on $L^2(m)$, and hence it is associated with a closed quadratic form, say $(\bar{\mathscr{E}}_q, \bar{\mathscr{F}})$. It is easy to check that $\mathscr{F} \subset \bar{\mathscr{F}}$ by the approximating form. Hence we have an inclusion chain: $U^{q-A}(L^2(m)) = D(\bar{H}) \subset \mathscr{F}^{-A} \subset \mathscr{F} \subset \bar{\mathscr{F}}$. Now we will show that the restriction of $\bar{\mathscr{E}}_q$ on $\mathscr{F}^{-A}$ is nothing but $\mathscr{E}_q^{-A}$. By Lemma 6.3.3(b), we know that $U^q(L^2(m)) \subset \mathscr{F} \cap L^2(\rho_A) = \mathscr{F}^{-A} \subset \bar{\mathscr{F}}$. Let $f \in L^2(m)$ nonnegative and $g := U^q f$. A switching order of integration gives for $p > 0$

$$\begin{aligned} pU^{p+q-A}g &= \mathsf{P}^{\cdot} \int_0^\infty e^{-qt} f(X_t)(p \int_0^t e^{-ps}(\mathrm{Exp}A)_s ds)dt \\ &\leq \mathsf{P}^{\cdot} \int_0^\infty e^{-qt} f(X_t)(\mathrm{Exp}A)_t dt. \end{aligned}$$

Hence we conclude that

$$pU^{p+q-A}g \leq U^{q-A}f \quad \text{and} \quad pU^{p+q-A}g \longrightarrow g \text{ q.e.}$$

Now we have

$$\begin{aligned} p(g, g - pU^{p+q-A}g) &= p(g, g - pU^{p+q}g) - p^2(g, U_A^{p+q}U^{p+q-A}g) \\ &= p(g, g - pU^{p+q}g) - \nu_A(pU^{p+q}g \otimes pU^{p+q-A}g). \end{aligned}$$

Since $0 \leq pU^{p+q}g \otimes pU^{p+q-A}g \leq g \otimes U^{q-A}f$ and $g, U^{q-A}f \in L^2(\rho_A)$, it follows from the dominated convergence theorem that $\bar{\mathscr{E}}_q(g,g) = \mathscr{E}_q^{-A}(g,g)$; namely the restriction of $\bar{\mathscr{E}}_q$ on $U^q(L^2(m))$ is $\mathscr{E}_q^{-A}$. A consequence is that for all $g \in U^q(L^2(m))$,

$$\nu_A(|g| \otimes |g|) \leq \mathscr{E}_q(|g|, |g|) \leq \mathscr{E}_q(g, g). \tag{6.3.8}$$

For any $g \in \mathscr{F}^{-A} \subset \mathscr{F}$, there exists a sequence $\{g_n\} \subset U^q(L^2(m))$ which converges to g in $\mathscr{E}_q$-norm. By (6.3.8) $\{g_n\}$ is an $\bar{\mathscr{E}}_q$-Cauchy sequence and $\{g_n\}$ converges to g in $\bar{\mathscr{E}}_q$-norm. Consequently $\bar{\mathscr{E}}_q(g,g) = \mathscr{E}_q^{-A}(g,g)$ and the restriction of $\bar{\mathscr{E}}_q$ on $\mathscr{F}^{-A}$ coincides with $\mathscr{E}_q^{-A}$. Hence $(\mathscr{E}^{-A}, \mathscr{F}^{-A})$ is closable and for $u \in \mathscr{F}^{-A}$,

$$\nu_A(|u| \otimes |u|) \leq \mathscr{E}_q(u, u). \tag{6.3.9}$$

(d) $\mathscr{F}^{-A}$ is dense in $\mathscr{F}$ with $\mathscr{E}_q$-norm since it contains $U^q(L^2(m))$ by Lemma 6.3.3(b). Let $u \in p\mathscr{F}$. there exists a sequence $\{u_n\} \subset p\mathscr{F}^{-A}$ such that $u_n \to u$ in $\mathscr{E}_q$-norm. By (6.3.9) and using Fatou's lemma,

$$\nu_A(u \otimes u) \leq \text{liminf}_n \nu_A(u_n \otimes u_n) \leq \mathscr{E}_q(u, u). \tag{6.3.10}$$

However (6.3.10) holds for any $A \in \mathscr{P}_K$ (with a different q) and certainly holds for A^*. Hence

$$\rho_A(u^2) = \rho_{A^*}(u^2) = \nu_{A^*}(u \otimes u) \leq \mathscr{E}_q(u, u),$$

i.e., $\mathscr{F}^{-A} = \mathscr{F} \cap L^2(\rho_A) = \mathscr{F}$.

(e) Since $2k(A) < 1$, $A, 2A \in \mathscr{P}_K$ and we may choose s large enough such that for $u \in \mathscr{F} = \mathscr{F}^{-A}$

$$\begin{aligned} 0 &\leq \mathscr{E}_q(u, u) \\ &\leq \mathscr{E}_s(u, u) - 2\nu_A(u \otimes u) + \mathscr{E}_q(u, u) \\ &= \mathscr{E}_s^{-A}(u, u) + \mathscr{E}_q^{-A}(u, u) \\ &= \bar{\mathscr{E}}_s(u, u) + \bar{\mathscr{E}}_q(u, u). \end{aligned}$$

Since $\mathscr{F}$ is dense in $\bar{\mathscr{F}}$ in $\bar{\mathscr{E}}_q$-norm, for any $w \in \bar{\mathscr{F}}$, there exists $\{w_n\} \subset \mathscr{F}$ such that $w_n \to w$ in $\bar{\mathscr{E}}_q$-norm, then in $\bar{\mathscr{E}}_s$-norm. By the inequality above, $\{w_n\}$ is a $\mathscr{E}$-Cauchy sequence and $w_n \to w$ in L^2. Hence $w \in \mathscr{F}$; namely $\mathscr{F} = \bar{\mathscr{F}}$. That completes the proof. □

6.4 Examples

In this section we shall use Lévy processes to construct two examples.

Example 1. Let X be a symmetric Lévy process on R^d with its Lévy exponent

$$\phi(x) = \frac{1}{2}(Sx, x) + \frac{1}{2}\int (1 - \cos(x, y))J(dy), \tag{6.4.1}$$

where S is a $d \times d$ nonnegative definite matrix and J a symmetric measure on R^d carried by $\mathrm{R}^d - \{0\}$ satisfying $\int (1 \wedge |x|^2)J(dx) < \infty$. Then the corresponding Dirichlet form is

$$\mathscr{F} = \{u \in L^2 : \int \phi|\hat{u}(x)|^2 dx < \infty\}, \tag{6.4.2}$$
$$\mathscr{E}(u,u) = \int \phi(x)|\hat{u}(x)|^2 dx.$$

Let h a positive symmetric function on R^d with $h(0) = 0$, and $A_t := \sum_{s\le t} h(\Delta X_s)$. Then $(\mathrm{Exp}\, A)_t = \prod_{s\le t}(1 + h(\Delta X_s))$ and $A \in \mathscr{P}_K$ if and only if h is J-integrable. We may see that the form defined below is a lower semibounded closed quadratic form associated with the perturbation by $((\mathrm{Exp}\, A)_t)$:

$$\mathscr{F}^h := \mathscr{F}; \tag{6.4.3}$$
$$\mathscr{E}^h(u,u) := \mathscr{E}(u,u) - \int\int u(x+y)u(x)dxh(y)J(dy).$$

and the perturbation semigroup is still spatially homogeneous with Lévy exponent

$$\phi^h(x) = \frac{1}{2}(Sx,x) + \frac{1}{2}\int(1-\cos(x,y))(1+h(y))J(dy) - J(h). \tag{6.4.4}$$

In the case that X is a symmetric stable process of index $\alpha \in]0,2[$, the condition means

$$\int \frac{h(x)}{|x|^{d+\alpha}} dx < \infty. \tag{6.4.5}$$

The following example shows that we can not expect the Khas'minskii's lemma holds for the natural exponential function.

Example 2. Let X be a Lévy process on Z, the set of integers, with convolution semigroup π given by

$$\pi_t = e^{-t}\sum_n \frac{t^n}{n!} J^{*n},$$

where J is a probability measure on Z defined by $J(\{-n\}) = J(\{n\}) := \frac{c}{n^2}$ for $n \ge 1$ and $J(\{0\}) := 0$. Let h be a function on Z defined by $h(n) := \log|n|$

for $n \neq 0$ and $h(0) = 0$. Set $A_t := \sum_{s \le t} h(\Delta X_s)$, which is an AF of the Kato class since $J(h) < \infty$. We claim that $\mathsf{E}e^{A_t} = \infty$ for any $t > 0$. (We write P^x as P since $\mathsf{P}^x e^{A_t}$ does not depend on x while A is spatially homogeneous.)

Suppose that $\mathsf{P}e^{A_t} < \infty$. Then there exist constants $c, q > 0$ such that for all $s > 0$

$$\mathsf{P}e^{A_s} \le ce^{qt}.$$

Hence $e^{-qs}\mathsf{P}e^{A_s} \le c$ and

$$\begin{aligned}
\mathsf{P}\sum_{s \le t} e^{-qs}(e^{\Delta A_s} - 1) &= \mathsf{P}\int_0^t e^{-qt}\frac{de^{A_s}}{e^{A_{s-}}} \\
&\le \mathsf{P}\int_0^t e^{-qs} dA^{A_s} \\
&= e^{-qt}\mathsf{E}e^{A_t} - 1 + q\int_0^t e^{-qs}\mathsf{P}e^{A_s} ds \le c - 1 + qtc < \infty.
\end{aligned}$$

By Lévy system formula,

$$\mathsf{P}\sum_{s \le t} e^{-qs}(e^{\Delta A_s} - 1) = J(e^h - 1)\int_0^t e^{-qs} ds.$$

This leads to a contradiction since $e^{h(n)} - 1 = |n| - 1$ for $n \neq 0$ and $J(e^h - 1) = \infty$.

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