Introduction to Markov Processes

Jiangang Ying
Fudan University

June 20, 2011

Contents

Introduction

I have planned for years and am now trying hard to write a book on theory of Markov processes and symmetric Markov processes so that graduate students in this field can move to the frontier quickly. In my impression, Markov processes are very intuitive to understand and manipulate. However to make the theory rigorously, one needs to read a lot of materials and check numerous measurability details it involved. This is a kind of dirty work a fresh graduate student hates to do, but has to do. My purpose is to help young researchers who are interested in this field to reduce the fear when they face it.

This book roughly covers materials of general theory of Markov processes, probabilistic potential theory, Dirichlet forms and symmetric Markov processes. I dare not say that all results are stated and proven rigorously, but I could say main ideas are included. For completeness and rigorousness, the readers may need to consult other books. The classic reference books are listed as follows.

1. Dynkin, Markov processes, Moscow, 1963; English translation, Springer, Berlin, 1965

2. Kellogg, Foundations of potential theory, Springer, 1967

3. Meyer, Processus de Markov, Lecture Notes in Math 26, Springer, 1967

4. Blumenthal, Getoor, Markov processes and potential theory, Academic Press, 1968

5. Getoor, Markov processes: ray processes and right processes, Lecture Notes in Math 440, Springer, 1975

6. Berg & Forst, Potential theory on locally compact abelian group, Springer, 1975

7. Port and Stone, Brownian motion and classical potential theory, Academic Press, 1978

8. Fukushima, Dirichlet forms and Markov processes, Kodansha and North-Holland, 1980

9. Chung, From Markov processes to Brownian motion, Springer, 1982

10. Doob, Classical potential theory and its probabilistic counterpart, Springer, 1983

11. Bliedtner & Hansen, Potential Theory, Springer, 1986

12. Sharpe, General theory of Markov processes, Academic Press, 1988

13. Getoor, Excessive measures, Birkhauser, 1990

14. Dellacherie, Meyer, Probabilites et Potentiels, Hermann, 1978-1992

15. Ma, Röckner, Introduction to the theory of Dirichlet forms, Springer, 1992

16. Fukushima, Oshima, Takeda, Dirichlet forms and symmetric Markov processes, de Gruyter, 1995

We shall state some fundamental results in general theory of stochastic processes mainly developed by Strasburg school of probability. Let $(\Omega, \mathscr{F}, \mathsf{P})$ be a complete probability space. A family of σ-algebra $(\mathscr{F}_t) = \{\mathscr{F}_t : t \geq 0\}$ is called a **filtration** if for any $0 \leq s < t$, $\mathscr{F}_s \subset \mathscr{F}_t \subset \mathscr{F}$. We say a filtration $(\mathscr{F}_t)$ satisfies the **usual condition** if each $\mathscr{F}_t$ contains all null sets in $\mathscr{F}$ and it is right continuous, namely, for each $t \geq 0$,

$$\mathscr{F}_t = \mathscr{F}_{t+} = \bigcap_{s>t} \mathscr{F}_s. \tag{0.1}$$

Let $(\mathscr{F}_t)$ be a filtration and $X = (X_t : t \geq 0)$ a real-valued stochastic process. X is $(\mathscr{F}_t)$-**adapted** (or adapted, if no confusion will be caused) if for each $t \geq 0$, X_t is $\mathscr{F}_t$-measurable. Moreover X is $(\mathscr{F}_t)$-**progressively measurable** if for each $t \geq 0$, the map $(s, \omega) \mapsto X_s(\omega)$ is measurable as a map from $([0,t] \times \Omega, \mathscr{B}[0,t] \times \mathscr{F}_t)$ to $(\mathbf{R}, \mathscr{B}(\mathbf{R}))$, where $\mathscr{B}[0,t]$ and $\mathscr{B}(\mathbf{R})$ are Borel σ-algebra on $[0,t]$ and $\mathbf{R}$, respectively. If no confusion is caused, $(\mathscr{F}_t)$ in the front may be omitted. A subset $A \subset \mathbf{R} \times \Omega$ is progressively measurable if so is the process $(t, \omega) \mapsto 1_A(t, \omega)$. A process is called right continuous or continuous or left continuous if almost all sample path has such regularity.

Theorem 0.0.1 A right continuous and adapted process is progressively measurable.

The least σ-algebra on $\mathbf{R} \times \Omega$ such that all adapted right continuous real processes are measurable is denoted by $\mathscr{O}$, an optional σ-field. A process which is $\mathscr{O}$-measurable is called **optional**. Then the theorem above implies that an optional process is progressively measurable. A map $\tau : \Omega \mapsto [0, \infty]$ is called an $(\mathscr{F}_t)$-stopping time if for each $t \geq 0$, $\{\tau \leq t\} \in \mathscr{F}_t$. For a stochastic process X and a subset $A \subset \mathbf{R}$, define the hitting time of A as

$$T_A = \inf\{t > 0 : X_t \in A\}. \tag{0.2}$$

Theorem 0.0.2 If the filtration $(\mathscr{F}_t)$ satisfies the usual condition and X is progressively measurable, then for any Borel subset $A \subset \mathbf{R}$, T_A is a stopping time.

The following theorem is called the section theorem, which is fundamental.

t:section **Theorem 0.0.3** Let X be a bounded progressively measurable process. If for any bounded decreasing sequence of stopping times $\{T_n\}$,

$$\lim_n \mathsf{E}[X_{T_n}] = \mathsf{E}[X_{\lim_n T_n}], \tag{0.3}$$

then X is right continuous.

Chapter 1

Right Markov processes

1.1 Right continuous Markov processes

In this section we shall first introduce the notion of right processes, which, essentially due to P.A. Meyer, makes classical potential theory operate almost naturally on it. Though, more or less, right processes are right continuous Markov processes with strong Markov property, it is a difficult task to give the definition clearly and concisely. Let $(E, \mathscr{E})$ be a topological space with its Borel σ-algebra. For any probability measure μ on E, $\mathscr{E}^\mu$ is the completion of $\mathscr{E}$ under μ and set

$$\mathscr{E}^* = \bigcap_\mu \mathscr{E}^\mu \tag{1.1}$$

where μ runs over all probability measures on E. A set in $\mathscr{E}^*$ is called a universally measurable subset of E. Any probability measure on $(E, \mathscr{E})$ may be uniquely extended on $\mathscr{E}^*$. The requirement for topology on E may vary, but in most cases, Radon space or Lusin space, which is a universally measurable subset or Borel subset of a compact metric space, respectively. One reason why we need to start from seemly so general topology is that in this way the class of right processes keeps stable under usual transformation in Markov processes such as killing transform, time change and drift transform.

Definition 1.1.1 Let E be a Radon space. A family of kernels $(P_t)_{t\geq 0}$ on $(E, \mathscr{E}^*)$ is called a transition semigroup if $P_t P_s = P_{t+s}$ for any $t, s \geq 0$ and $P_t(x, E) \leq 1$ for any $t \geq 0$ and $x \in E$. In addition, if $P_t(x, E) = 1$ for any $t \geq 0$ and $x \in E$, it is called a transition probability semigroup. A transition semigroup (P_t) is called a Borel semigroup if E is Lusin space and each P_t is a kernel on $(E, \mathscr{E})$, or maps a Borel measurable function to a Borel measurable function.

It is known that by joining a point Δ, called a cemetery point, to E, (P_t) may be extended into a transition probability semigroup on E_Δ.

Definition 1.1.2 Let $(E, \mathscr{E}^*)$ be a Radon space and its universal Borel sunsets. A group of notations

$$X = (\Omega, \mathscr{G}, \mathscr{G}_t, X_t, \theta_t, \mathsf{P}^x)$$

is called a **right continuous Markov process** on state space E or say X satisfies (HD1) if the following conditions are satisfied.

(1) $(\Omega, \mathscr{G}, \mathscr{G}_t)$ is a filtered measurable space and (X_t) is an E_Δ-valued process $\mathscr{E}^*_\Delta$-adapted to $(\mathscr{G}_t)$, more precisely for any $t \geq 0$, X_t is a measurable mapping from $(\Omega, \mathscr{G}_t)$ to $(E_\Delta, \mathscr{E}^*_\Delta)$. For every $x \in E_\Delta$, P^x is a probability measure on $(\Omega, \mathscr{G})$

(2) $(\theta_t)_{t\geq 0}$ is a family of shift operators for X, i.e., $\theta_t : \Omega \to \Omega$ and, identically for any $t, s \geq 0$,

$$\theta_t \circ \theta_s = \theta_{t+s} \text{ and } X_t \circ \theta_s = X_{t+s}.$$

(3) (normality) For $x \in E$,

$$\mathsf{P}^x(X_0 = x) = 1.$$

Moreover $x \mapsto \mathsf{P}^x(H)$ is universally measurable for any $H \in \mathscr{G}$.

(4) (Markov property) For every $t, s \geq 0$, $f \in b\mathscr{E}^*$ and $x \in E$, it holds P^x-a.s.

$$\mathsf{P}^x(f(X_{t+s})|\mathscr{G}_s) = P_t f(X_s). \tag{1.2}$$

(5) (regularity) $t \mapsto X_t$ is a right continuous process on $E_\Delta = E \cup \{\Delta\}$ almost surely.

(6) (life time) Define $\zeta(\omega) := \inf\{t : X_t = \Delta\}$. Then $X_t(\omega) \in E$ for $t < \zeta(\omega)$, and $X_t(\omega) = \Delta$ for all $t \geq \zeta(\omega)$. Hence ζ is called the lifetime of X.

A transition semigroup (P_t) satisfies (HD1) if it has a realization satisfying (HD1). The word **for almost all** means "for any $x \in E$ and P^x-almost all", i.e., a measurable subset Ω_0 of Ω such that $\mathsf{P}^x(\Omega_0) = 1$ for all $x \in E$. Notice that the measurability in (3) is not so much restricted since it holds at least for σ-algebra generated by (X_t) itself. Let $(\mathscr{F}_t^{0*})$ (resp., $(\mathscr{F}_t^0)$) be the natural filtration of (X_t) generated by $\mathscr{E}^*$ (resp., $\mathscr{E}$), precisely,

$$\mathscr{F}_t^{0*} = \sigma\left(\bigcup_{s \leq t} X_s^{-1}(\mathscr{E}^*)\right), \ \mathscr{F}_t^0 = \sigma\left(\bigcup_{s \leq t} X_s^{-1}(\mathscr{E})\right). \tag{1.3}$$

Clearly for any $t \geq 0$, $\mathscr{F}_t^{0*} \subset \mathscr{G}_t$ and $\mathscr{F}_\infty^{0*} \subset \mathscr{G}$. By monotone class theorem, $x \mapsto \mathsf{P}^x(H)$ is $\mathscr{E}^*$-measurable for any $H \in \mathscr{F}_\infty^{0*}$. Furthermore if (P_t) is Borel semigroup, $x \mapsto \mathsf{P}^x(H)$ is $\mathscr{E}$-measurable for any $H \in \mathscr{F}_\infty^0$.

Fix now such a process X on E. For any probability measure μ on $(E, \mathscr{E})$, define

$$\mathsf{P}^\mu(H) = \int_E \mathsf{P}^x(H)\mu(dx), \ H \in \mathscr{G}. \tag{1.4}$$

Denote by $\mathscr{G}^\mu$ the completion of $\mathscr{G}$ with respect to P^μ and $\mathscr{G}_t^\mu$ the σ-field generated by $\mathscr{G}_t$ and all P^μ-null sets in $\mathscr{G}^\mu$. We usually say that $\mathscr{G}_t^\mu$ is the **augmentation** of $\mathscr{G}_t$ in $(\Omega, \mathscr{G}, \mathsf{P}^\mu)$.

Exercise 1.1 Prove that the completion of $\mathscr{F}_\infty^0$ with respect to P^μ is equal to the completion of $\mathscr{F}_\infty^{0*}$ with respect to P^μ. The same conclusion holds for the augmentation of $(\mathscr{F}_t^0)$ and $(\mathscr{F}_t^{0*})$ in $(\Omega, \mathscr{F}, \mathsf{P}^\mu)$.

Set

$$\widetilde{\mathscr{G}} = \bigcap_\mu \mathscr{G}^\mu, \widetilde{\mathscr{G}_t} = \bigcap_\mu \mathscr{G}_t^\mu, \tag{1.5}$$

where μ runs over all probability measures on $(E, \mathscr{E})$. The filtration $(\widetilde{\mathscr{G}_t})$ is called the augmentation of $(\mathscr{G}_t)$. It is not hard to see that the process has Markov property with respect to $(\widetilde{\mathscr{G}_t})$ and actually for any probability measure μ on E, it holds P^μ-a.s. for $t, s \geq 0$, $f \in b\mathscr{E}^*$

$$\mathsf{P}^\mu(f(X_{t+s})|\mathscr{G}_s^\mu) = P_t f(X_s). \tag{1.6}$$

The procedure to get $(\widetilde{\mathscr{G}_t^\mu})$ and $(\widetilde{\mathscr{G}_t})$ is called **augmentation** of the filtration of X with respect to the laws (P^x). This is a 'dirty' work which has to be done for a Markov process. Therefore we may assume from the beginning that $\mathscr{G}$ and $(\mathscr{G}_t)$ are augmented. The augmentation of the natural filtration $(\mathscr{F}_t^0)$ is denoted by $(\mathscr{F}_t)$, which is also the augmentation of $(\mathscr{F}_t^{0*})$. After the augmentation, we have to check that we still have the necessary measurability such as

(1) For $B \in \mathscr{G}$, $x \mapsto \mathsf{P}^x(B)$ is universally measurable;

(2) X_t is measurable from $(\Omega, \mathscr{G}_t)$ to $(E, \mathscr{E}^*)$;

(3) θ_t is measurable on $(\Omega, \mathscr{G})$.

The good news about augmentation which we shall prove later is that $(\mathscr{G}_t)$ will satisfy the usual condition when a slight more condition is imposed, and then the hitting time of any optional set is then a stopping time.

For $\alpha > 0$, a $[0, \infty]$-valued measurable function f on $(E, \mathscr{E}^*)$ is α-supermidian if $e^{-\alpha t} P_t f \leq f$ for each $t > 0$ and α-excessive if, in addition, $e^{-\alpha t} P_t f \uparrow f$ as $t \downarrow 0$. Let $\mathbf{S}^\alpha$ be the set of all α-excessive functions.

Definition 1.1.3 Let E be a Radon space and (P_t) a transition semigroup on E. Assume that the collection

$$X = (\Omega, \mathscr{G}, \mathscr{G}_t, X_t, \theta_t, \mathsf{P}^x)$$

is a right continuous Markov process on E with (P_t) as its semigroup. Then X is said to be a **right process** provided X satisfies (HD2), namely for any α-excessive function f, $t \mapsto f(X_t)$ is right continuous almost surely. Moreover if E is Lusin space and (P_t) is a Borel semigroup, then X is called a Borel right process.

The first important property of right processes is strong Markov property. We now give two fundamental theorems for right processes. Note that we may always assume that Ω is the canonical space, i.e., the space of right continuous maps from $[0, \infty)$ to E. To state strong Markovian property, we assume that readers are familiar with the theory related to stopping times.

We shall now introduce the notion of potential which plays an essential role in general theory of Markov processes. To define α-potentials, some measurability needs to be clarified in advance. For a bounded continuous function f on E, $t \mapsto f(X_t)$ is right continuous and hence $(t, x) \mapsto \mathsf{E}^x[f(X_t)] = P_t f(x)$

jointly measurable on $(\mathbf{R}^+ \times E, \mathscr{B}(\mathbf{R}^+) \times \mathscr{E}^*)$. It is also true for bounded Borel measurable f by monotone class theorem. The following exercise makes it possible to define resolvent of (P_t) and use Fubini theorem.

Exercise 1.2 For $f \in b\mathscr{E}^*$, $(t,x) \mapsto \mathsf{E}^x[f(X_t)]$ is measurable for the completion of $(\mathscr{B}(\mathbf{R}^+) \times \mathscr{E}^*)$ with respect to the product measure of any finite measure on $\mathbf{R}^+$ and a finite measure on $\mathscr{E}^*$.

For $\alpha > 0$ and $f \in b\mathscr{E}^*$, define the resolvent or α-potential of f

$$U^\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x) dt \tag{1.7}$$

$$= \mathsf{E}^x \int_0^\infty e^{-\alpha t} f(X_t) dt. \tag{1.8}$$

Then we have the well-known resolvent equation

$$U^\alpha = U^\gamma + (\gamma - \alpha) U^\alpha U^\gamma \tag{1.9}$$

for $\alpha, \gamma > 0$.

t:100425-1 **Theorem 1.1.4** Let X be a right process on E with transition semigroup (P_t). Then

(1) X has strong Markov property with respect to $(\mathscr{F}^0_{t+})$, i.e., for any $(\mathscr{F}^0_{t+})$-stopping time σ, $f \in b\mathscr{E}^*$, $t > 0$ and $x \in E$,

$$\mathsf{E}^x[f(X_{t+\sigma}) 1_{\{\sigma<\infty\}} | \mathscr{F}^0_{\sigma+}] = 1_{\{\sigma<\infty\}} \mathsf{E}^{X_\sigma}[f(X_t)], \tag{1.10}$$

P^x-a.s.;

(2) for any probability μ on E, $(\mathscr{F}^\mu_t)$ is right continuous, and then $(\mathscr{F}_t)$ is right continuous.

Proof. (1) Let f be a uniformly continuous bounded function on E. Assume that $\sigma < \infty$. Set

$$\sigma_n = \sum_{k\geq 1} \frac{k}{2^n} 1_{\{(k-1)2^{-n}\leq\sigma<k2^{-n}\}}$$

Then $\sigma_n \downarrow \sigma$ and σ_n is $(\mathscr{F}_t^0)$-stopping time. By the right continuity and simple Markov property of X, we have for any probability μ on E,

$$\begin{aligned}
&\mathsf{E}^\mu \int_0^\infty \mathrm{e}^{-\alpha t} f(X_{t+\sigma})\mathrm{d}t = \lim_n \mathsf{E}^\mu \int_0^\infty \mathrm{e}^{-\alpha t} f(X_{t+\sigma_n})\mathrm{d}t \\
&= \lim_n \sum_k \mathsf{E}^\mu \left(\int_0^\infty \mathrm{e}^{-\alpha t} f(X_{t+\frac{k}{2^n}})\mathrm{d}t; \sigma_n = \frac{k}{2^n} \right) \\
&= \lim_n \sum_k \mathsf{E}^\mu \left(\mathsf{E}^{X(\frac{k}{2^n})} \int_0^\infty \mathrm{e}^{-\alpha t} f(X_t)\mathrm{d}t; \sigma_n = \frac{k}{2^n} \right) \\
&= \lim_n \mathsf{E}^\mu U^\alpha f(X_{\sigma_n}) = \mathsf{E}^\mu U^\alpha f(X_\sigma) \\
&= \int_0^\infty \mathrm{e}^{-\alpha t} \mathsf{E}^\mu \mathsf{E}^{X(\sigma)} f(X_t)\mathrm{d}t.
\end{aligned}$$

This means that $t \mapsto \mathsf{E}^\mu f(X_{t+\sigma})$ and $t \mapsto \mathsf{E}^\mu[\mathsf{E}^{X(\sigma)} f(X_t)]$ have the same Laplace transform and it implies they are identical because they are both right continuous. Hence

$$\mathsf{E}^\mu f(X_{t+\sigma}) = \mathsf{E}^\mu[\mathsf{E}^{X(\sigma)} f(X_t)]$$

from which, the strong Markov property with respect to $(\mathscr{F}_{t+}^0)$ follows.

(2) Obviously (1) implies that X has simple Markov property with respect to $(\mathscr{F}_{t+}^0)$, i.e., for any bounded random variable Y on $(\Omega, \mathscr{F}_\infty^0)$ and a probability μ on E, it holds P^μ-a.s.

$$\mathsf{E}^\mu(Y \circ \theta_t | \mathscr{F}_{t+}^0) = \mathsf{E}^{X_t}(Y) = \mathsf{E}^\mu(Y \circ \theta_t | \mathscr{F}_t^0). \tag{1.11}$$

It is easy to verify that when $Y = f_1(X_1) \cdots f_n(X_{t_n})$,

$$\mathsf{E}^\mu(Y | \mathscr{F}_{t+}^0) = \mathsf{E}^\mu(Y | \mathscr{F}_t^0),$$

and actually it holds for any $Y \in b\mathscr{F}^0_\infty$ by monotone class theorem. Then for $A \in \mathscr{F}^0_{t+}$, we have P^μ-a.s. $1_A = \mathsf{E}^\mu(1_A|\mathscr{F}^0_t)$, and hence $A \in \mathscr{F}^\mu_t$. It implies that

$$\mathscr{F}^0_{t+} \subset \mathscr{F}^\mu_t.$$

The conclusion follows from an assertion that the σ-field generated by $\mathscr{F}^0_{t+}$ and P^μ-null sets equals $\mathscr{F}^\mu_{t+}$, which is left to the readers as an exercise. □

By (2) in Theorem t:100425-1 1.1.4, we may always assume without loss of generality that the filtration $(\mathscr{G}_t)$ satisfies usual condition, i.e., it contains all null sets and right continuous. By augmentation, the strong Markov property may be stated as follows. For any probability μ and a non-negative function $f \in \mathscr{E}^*$, if σ is an $(\mathscr{F}^\mu_t)$-stopping time, then

$$\mathsf{E}^\mu(f(X_{t+\sigma})1_{\{\sigma<\infty\}}|\mathscr{F}^\mu_t) = P_t f(X_\sigma)1_{\{T<\infty\}}. \tag{1.12}$$

Exercise 1.3 Prove that

$$\mathsf{E}^\mu \int_T^\infty e^{-\alpha t} f(X_t)\mathrm{d}t = \mathsf{E}^\mu\left(e^{-\alpha T}U^\alpha f(X_T)\right). \tag{1.13}$$

The following lemma lists some properties of excessive functions and is easy to verify. For $\alpha \geq 0$, a non-negative measurable function f on E, which may take infinity, is called α-excessive, write $f \in \mathbf{S}^\alpha$, if (1) $e^{-\alpha t}P_t f \leq f$ for each $t > 0$; (2) $e^{-\alpha t}P_t f$ converges to f as $t \downarrow 0$. When $\alpha = 0$, we simply say f is excessive and $f \in \mathbf{S}$.

l:100428-1 **Lemma 1.1.5** (1) $\mathbf{S}^\alpha$ is a cone.

(2) $\mathbf{S}^\alpha$ is stable under increasing limit.

(3) If $\alpha > \beta \geq 0$, $\mathbf{S}^\alpha \supset \mathbf{S}^\beta$ and $\mathbf{S}^\beta = \bigcap_{r>\beta} \mathbf{S}^r$.

(4) If $f, g \in \mathbf{S}^\alpha$, $f \wedge g \in \mathbf{S}^\alpha$.

(5) If f is α-super-median and μ is a probability measure on E satisfying

$\mu(f) < \infty$, then the process $(e^{-\alpha t} f(X_t))$ is a super-martingale with respect to P^μ.

Actually the proof of (4) needs to use (HD2) on the process.

l:100428-2 **Lemma 1.1.6** (1) For $\alpha \geq 0$ and $\mathscr{E}^*$-measurable $f \geq 0$, $U^\alpha f \in \mathbf{S}^\alpha$.
(2) For $\alpha \geq 0$, $f \in \mathbf{S}^\alpha$ if and only if $\beta U^{\alpha+\beta} f \uparrow f$ as $\beta \uparrow +\infty$.
(3) For $\alpha > 0$ and $f \in \mathbf{S}^\alpha$, there exist $g_n \in b\mathscr{E}^*_+$ such that $U^\alpha g_n \uparrow f$ as $n \uparrow +\infty$.

Since a super-martingale which is the limit of a sequence of right continuous super-martingales is also right continuous, we shall state following weaker forms of (HD2). A negative function f on E is nearly Borel for X if for each probability μ on E there exist $f_1, f_2 \in \mathscr{E}$ with $f_1 \leq f \leq f_2$ such that two processes $(f_1(X_t))$ and $(f_2(X_t))$ are P^μ-indistinguishable. A measurable function f on $(E, \mathscr{E}^*)$ is called **optional** if $t \mapsto f(X_t)$ is indistinguishable from an $(\mathscr{F}_t)$-optional process, and **nearly optional** if for any probability measure μ on E, $t \mapsto f(X_t)$ is indistinguishable from an $(\mathscr{F}^\mu_t)$-optional process. A set $A \in \mathscr{E}^*$ is optional or nearly optional if so is 1_A. Let $\mathscr{E}^{no}$ be the set of nearly optional subsets of E which is a σ-algebra.

Exercise 1.4 Prove that f is nearly optional if f is $\mathscr{E}^{no}$-measurable.

The next theorem follows from the section theorem as stated in Theorem t:section 0.0.3.

t:100425-2 **Theorem 1.1.7** Assume that (HD1) holds. If X is strong Markov and each α-excessive function is nearly optional, then (HD2) holds. Therefore if (P_t) is Borel and X has strong Markov property, then (HD2) holds and each α-excessive function is nearly Borel.

Proof. For any probability μ, an increasing sequence $\{T_n\}$ of stopping times with $T = \lim T_n$, and a non-negative bounded function $g \in \mathscr{E}^*$, we have by strong Markov property

$$\mathsf{E}^\mu \left(e^{-\alpha T_n} U^\alpha g(X_{T_n}) \right) = \mathsf{E}^\mu \int_{T_n}^\infty e^{-\alpha t} g(X_t) \mathrm{d}t$$
$$\to \mathsf{E}^\mu \int_T^\infty e^{-\alpha t} g(X_t) \mathrm{d}t = \mathsf{E}^\mu \left(e^{-\alpha T} U^\alpha g(X_T) \right).$$

Combining the assumption that $U^\alpha g$ is nearly optional, it follows that $t \mapsto U^\alpha g(X_t)$ is right continuous from Theorem t:section 0.0.3. Finally by Lemma l:100428-2 1.1.6(3), $t \mapsto f(X_t)$ is right continuous for any α-excessive function f. □

Remark 1.1.8 Though Theorem t:100425-2 1.1.7 hints that (HD2) may be equivalent to strong Markov property, an example, when (P_t) is not Borel, is presented by Salisbury to show that a right continuous Markov process with strong Markov property may not be a right process.

t:0514-1 **Theorem 1.1.9** Assume (HD1) holds. Let $\mathbf{C}$ be a linear subspace of $C(E)$, closed under function multiplication, which generates $\mathscr{E}$. If, for any bounded $f \in \mathbf{C}$, the process $t \mapsto U^\alpha f(X_t)$ is right continuous, then (HD2) holds.

Proof. It suffices to show that $U^\alpha g$ is nearly optional for non-negative and bounded $g \in \mathscr{E}^*$. It is true by monotone class theorem for $g \in \mathscr{E}$ and it follows from the proof of Theorem t:100425-2 1.1.7 that $t \mapsto U^\alpha g(X_t)$ is right continuous. Let now $g \in \mathscr{E}^*$ be bounded. For any probability μ on E, there exist $g_1, g_2 \in \mathscr{E}$ such that $g_1 \le g \le g_2$ and $\mu U^\alpha (g_2 - g_1) = 0$. Then for any $t > 0$, $U^\alpha g_1(X_t) \le U^\alpha g(X_t) \le U^\alpha g_2(X_t)$ and

$$\mathsf{E}^\mu (U^\alpha (g_2 - g_1)(X_t) = \mu P_t U^\alpha (g_2 - g_1) \le e^{\alpha t} \mu U^\alpha (g_2 - g_1) = 0.$$

Therefore two processes $U^\alpha g_2(X.)$ and $U^\alpha g_1(X.)$ are P^μ-distinguishable, i.e., $U^\alpha g$ is nearly optional. □

Example 1.1.10 (α-subprocess) Let X be a right process on E with transition semigroup (P_t). For $\alpha > 0$, it is known that $P_t^\alpha = e^{-\alpha t}P_t$ is also a transition semigroup on E. Is it a transition semigroup of a right process? Sure it is. But how do we construct the right process? Introduce the killing operators (k_t) on Ω:

$$X_s \circ k_t = \begin{cases} X_s, & s < t, \\ \Delta, & s \geq t, \end{cases} \tag{1.14}$$

Intuitively k_t makes no change before time t but sends the path after t to cemetery. For $x \in E$, define probability $\mathbb{Q}^x$ on $(\Omega, \mathscr{F})$ by

$$\mathbb{Q}^x(Y) = \mathsf{E}^x \int_0^\infty Y \circ k_u \mathrm{d}(-e^{-\alpha u}) = \alpha \mathsf{E}^x \int_0^\infty Y \circ k_u e^{-\alpha u} \mathrm{d}u, \tag{1.15}$$

where Y is a bounded or non-negative random variable on Ω. Note that we use $\mathbb{Q}$ for both probability and expectation. Let

$$X^\alpha = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, \mathbb{Q}^x)$$

which is called α-subprocess of X. It is easy to check that X is a right process with transition semigroup (P_t^α). In fact,

$$\begin{aligned} \mathbb{Q}^x(f(X_t)) &= \alpha \mathsf{E}^x \int_0^\infty f(X_t) \circ k_u e^{-\alpha u} \mathrm{d}u \\ &= \alpha \mathsf{E}^x \int_t^\infty f(X_t) e^{-\alpha u} \mathrm{d}u \\ &= e^{-\alpha t} \mathsf{E}^x(f(X_t)) = P_t^\alpha f(x). \end{aligned}$$

The verification of (HD2) is left for those who are interested. ∎

Example 1.1.11 (Killing at leaving) Let X be a right process on E with transition semigroup (P_t). Intuitively for a subset B, killing X at leaving B shall give us a process which certainly inherits Markov property from X.

Rigorously speaking, let $B \in \mathscr{E}^o$ and $T = T_B$ the hitting time of B. Define a map $\omega \mapsto k_T\omega$ on Ω by

$$k_T\omega(t) = \begin{cases} \omega(t), & t < T; \\ \Delta, & t \geq T. \end{cases} \tag{1.16}$$

Hence the new lifetime is $\zeta \wedge T$. ∎

Example 1.1.12 (Doob's h-transform) ∎

1.2 Feller processes and Lévy processes

A question we must ask is when and how we will have a right Markov process. There are basically two ways: one is from Feller semigroup and the other is through transformation as the example in the last section shows. In this section we shall introduce Feller semigroup and prove that it may be realized as a Markov process much better than a right process.

Definition 1.2.1 Let E be a locally compact metrizable space with a countable base. A transition semigroup (P_t) on E is called a Feller semigroup if
(1) $P_tC_\infty(E) \subset C_\infty(E)$ for each $t > 0$;
(2) for each $f \in C_\infty(E)$,

$$\lim_{t\downarrow 0} \|P_tf - f\|_\infty = 0.$$

With other conditions, (2) above is equivalent to a weaker one: for each $f \in C_\infty(E)$ and $x \in E$, $P_tf(x) \to f(x)$ as $t \downarrow 0$. The proof is a good exercise. Since $C_\infty(E)$ is a Banach space and Feller semigroup (P_t) is a strongly continuous semigroup on $C_\infty(E)$, its infinitesimal generator determines (P_t) completely by Hille-Yosida theorem. The following theorem is actually a corollary of regularization theorem of super-martingales.

Theorem 1.2.2 Let (P_t) be a Feller semigroup on E. Then (P_t) has a realization which is a Borel right process, which is called a **Feller process**.

Proof. Add a point Δ to E such that E_Δ is compact and (P_t) is extended to a probability transition semigroup on E_Δ. Any function f on E may be always viewed as a function on E_Δ by defining $f(\Delta) = 0$. In this way

$$C_\infty(E) = \{f \in C(E_\Delta) : f(\Delta) = 0\}.$$

Let $X = (X_t, \mathsf{P}^x)$ be a realization of (P_t) on E_Δ. For any non-negative $f \in C_\infty(E)$ and $\alpha > 0$, the process $(e^{-\alpha t} U^\alpha f(X_t) : t \geq 0)$ is a supermartingale with respect to P^x for each $x \in E$. It follows that $t \mapsto U^\alpha f(X_t)$ has right and left limits P^x-almost surely. We may take a countable subset D of $\{U^\alpha f : \alpha > 0, f \in C_\infty(E_\Delta)\}$ separating points in E. Since D is countable, there exists $N_0 \subset \Omega$ such that $\mathsf{P}^x(N_0) = 0$ for all $x \in E$ and for any $g \in D$ and $\omega \notin N_0$, $t \mapsto g(X_t(\omega))$ has right and left limits. From the facts that D separates points in E and any function in D is continuous, it follows that for $\omega \notin N_0$, $t \mapsto X_t(\omega)$ has right and left limits. Let $Y = (Y_t)$ is the right limit process of X, namely

$$Y_t(\omega) = \lim_{s \downarrow\downarrow t} X_t(\omega), \ t \geq 0, \omega \notin N_0.$$

It suffices to show that Y is a version of X. Fix $t \geq 0$ and $s > 0$. Take any non-negative functions $f, g \in C_\infty(E)$ and

$$\begin{aligned} \mathsf{E}^x(f(X_t)g(X_{s+t})) &= \mathsf{E}^x(f(X_t)P_s g(X_t)) \\ &= P_t(f P_s g)(x). \end{aligned}$$

As $s \downarrow 0$, $X_{s+t} \to Y_t$, $P_s g \to g$ and hence we have

$$\mathsf{E}^x(f(X_t)g(Y_t)) = P_t(fg)(x) = \mathsf{E}^x(f(X_t)g(X_t)).$$

It follows from the monotone convergence theorem that for any continuous function $h \geq 0$ on $E \times E$,

$$\mathsf{E}^x[h(X_t, Y_t)] = \mathsf{E}^x[h(X_t, X_t)]$$

and we have $X_t = Y_t$ a.s.

Hence $Y = (Y_t)$ is a right continuous realization of (P_t) and it is easy to see that Y is a Borel right process, due to Theorem t:0514-1 1.1.9 and the fact that $U^\alpha f$ is continuous for any $f \in C_\infty(E)$. □

An important example of Feller semigroup is the convolution semigroup on Euclidean space, whose right continuous realization is called a Lévy process.

Definition 1.2.3 A family of probability measures $\{\nu_t : t > 0\}$ on $\mathbf{R}^d$ is called a **convolution semigroup** if
(1) $\nu_t * \nu_s = \nu_{t+s}$ for any $t, s > 0$;
(2) $\nu_t \to \varepsilon_0$ weakly as $t \downarrow 0$ where ε_0 is the point mass at 0.

Let $\{\nu_t\}$ be a convolution semigroup on $\mathbf{R}^d$ and set $P_t(x, dy) = \nu_t(dy - x)$. Then (P_t) is a Feller semigroup on $\mathbf{R}^d$ and its right continuous realization is called a **Lévy process** on $\mathbf{R}^d$. Actually many well-known Markov processes such as Brownian motion, Poisson process, stable process, are Lévy processes. The law of a Lévy process is determined by its convolution semigroup, which is in turn determined by its so-called Lévy exponent.

Let $\widehat{\nu_t}$ denote the characteristic function of ν_t which is bounded and continuous on $\mathbf{R}^d$. There exists a complex-valued continuous function φ on $\mathbf{R}^d$ such that

$$\widehat{\nu_t} = \exp(-t\varphi), \tag{1.17}$$ e:0514-2

and this φ determines $\{\nu_t\}$ uniquely by the uniqueness of Fourier transform and called the Lévy exponent of $\{\nu_t\}$. Obviously $\varphi(0) = 0$ and it is well-known that φ has the following representation: for $x \in \mathbf{R}^d$,

$$\varphi(x) = i(a,x) + \frac{1}{2}(Sx,x) + \int_{\mathbf{R}^d} \left(1 - e^{i(x,y)} + \frac{i(x,y)}{1+|y|^2}\right) \pi(dy), \qquad (1.18)$$

where $a \in \mathbf{R}^d$, S is a $d \times d$ non-negative definite symmetric matrix, and π is a Radon measure on $\mathbf{R}^d \setminus \{0\}$ having the following integrability

$$\int_{\mathbf{R}^d} \frac{|y|^2}{1+|y|^2} \pi(dy) < \infty. \qquad (1.19)$$

The matrix S and measure π are uniquely determined. But the vector a depends on the way we write (1.18). Conversely given a function φ as in (1.18), there must be a unique convolution semigroup $\{\nu_t\}$ on $\mathbf{R}^d$ such that (1.17) holds. This characterization is the famous **Lévy-Khinchin formula**, which tells us that every character about a Lévy process may be retrieved from its Lévy exponent.

It is easy to verify that Lévy exponent of Brownian motion is $\varphi(x) = \frac{1}{2}|x|^2$. When π is a finite measure and

$$\varphi(x) = \int_{\mathbf{R}^d} (1 - e^{i(x,y)}) \pi(dy), \qquad (1.20)$$

the corresponding semigroup (resp., Lévy process) is called the compound Poisson semigroup (resp., compound Poisson process). In this case, let $\lambda = \pi(\mathbf{R}^d)$ and $\pi_0 = \lambda^{-1}\pi$. At each step, the process will stay freezing at a position x for an exponentially distributed time with parameter λ and then jump to somewhere according to distribution $\pi_0(\cdot - x)$.

For a Lévy process X on $\mathbf{R}^d$ with convolution semigroup $\{\nu_t\}$, the Lebesgue measure m is always an invariant measure for X, since it is easy to check that

$\int_{\mathbf{R}^d} m(dx)\nu_t(A-x) = m(A)$ for any Borel subset A. X is called **symmetric** if

$$\nu_t(-A) = \nu_t(A) \tag{1.21}$$

for any Borel subset A of $\mathbf{R}^d$. It can be seen that in this case

$$m(dx)\nu_t(dy-x) = m(dy)\nu_t(dx-y). \tag{1.22}$$

Clearly X is symmetric if and only if its Lévy exponent φ is real, i.e.,

$$\varphi(x) = \frac{1}{2}(Sx,x) + \int_{\mathbf{R}^d}(1-\cos(x,y))\pi(dy),\ x \in \mathbf{R}^d. \tag{1.23}$$

Theorem 1.2.4 If X is symmetric, then any Radon invariant measure of X is a multiple of Lebesgue measure if and only if its Lévy exponent φ has unique zero.

1.3 Fine topology and balayage

The Blumenthal 0-1 law is easy to prove but very important.

Theorem 1.3.1 (Blumenthal) For any $A \in \mathscr{F}_0$ and $x \in E$, $\mathsf{P}^x(A)$ is either zero or one.

Proof. For any probability μ on E, there exists $B \in \mathscr{F}_0^0$ such that $\mathsf{P}^\mu(A \triangle B) = 0$. By Markov property, $\mathsf{P}^\mu(\theta_0^{-1}A \triangle \theta_0^{-1}B) = 0$. Since $\theta_0^{-1}B = B$, $\mathsf{P}^\mu(\theta_0^{-1}A \triangle A) = 0$. Then by Markov property again, for $x \in E$,

$$\mathsf{P}^x(A) = \mathsf{P}^x(A \cap \theta_0^{-1}A) = \mathsf{E}^x[\mathsf{P}^{X_0}(A); A] = (\mathsf{P}^x(A))^2$$

and it follows that $\mathsf{P}^x(A) = 0$ or 1. □

If Blumethal 0-1 law was only talking about a set in $\mathscr{F}_0^0$, it would mean nothing. Its importance is due to the fact that $\mathscr{F}_0$ is much richer than $\mathscr{F}_0^0$.

Before we go any further we should answer a question: for what kind of subset B of E, the hitting time T_B is a stopping time for the augmented filtration $(\mathscr{F}_t)$? Let's start from two basic results. Given a filtration $(\mathscr{M}_t)$ and a measurable space $(S, \mathscr{S})$, an S-valued stochastic process (Y_t) is $(\mathscr{M}_t)$-**progressively measurable** if for any $t \geq 0$, $(s, \omega) \mapsto Y_s(\omega)$ is $\mathscr{B}([0,t]) \times \mathscr{M}_t/\mathscr{S}$-measurable.

Exercise 1.5 If Y is $(\mathscr{M}_t)$-progressively measurable and $\varphi: \ S \to \mathbf{R}$ is Borel measurable, then so is $t \mapsto \varphi \circ Y_t$.

For a set $A \subset E$, the **hitting time** and **entrance time** for A of X are $T_A = \inf\{t > 0 : X_t \in A\}$ and $D_A = \inf\{t \geq 0 : X_t \in A\}$. It is easy to see that T_A is a terminal time, i.e., almost surely $T_A \circ \theta_t + t = T_A$ on $\{T_A > t\}$ for all $t > 0$.

Lemma 1.3.2 (1) A right continuous and adapted process is progressively measurable. Therefore (X_t) is progressively measurable. (2) If the filtration satisfies the usual condition, the hitting time of a real progressively measurable process for a Borel set is a stopping time.

For $f \in \mathbf{S}^\alpha$, $t \mapsto f(X_t)$ is right continuous and so f is optional. Let $\mathscr{E}^e$ denote the σ-algebra generated by all excessive functions. Then

$$\mathscr{E} \subset \mathscr{E}^e \subset \mathscr{E}^{no} \subset \mathscr{E}^*. \tag{1.24}$$

Theorem 1.3.3 If A is nearly optional, then the hitting time T_A is an $(\mathscr{F}_t)$-stopping time.

Proof. By the definition and lemma above, T_A is an $(\mathscr{F}_t^\mu)$-stopping time for any probability μ on E and hence an $(\mathscr{F}_t)$-stopping time. □

For any $A \in \mathscr{E}^{no}$, since $\{T_A = 0\} \in \mathscr{F}_0$, $\mathsf{P}^x(T_A = 0) = 0$ or 1 for each $x \in E$, by Blumenthal 0-1 law. If it is zero, we say x is regular for A or otherwise x is irregular for A. Let A^r denote the set of regular points for A. A nearly optional subset G of E is called **finely open**, if for any $x \in G$, $\mathsf{P}^x(T_{G^c} = 0) = 0$ or equivalently x is irregular for G^c. Intuitively G is finely open if X, starting from any point in G, will not leave G immediately. It is routine to show that the set of finely open subsets in E is a topology, which we call the **fine topology** of X on E. Since X is right continuous, any point in an open subset G will not leave G immediately and hence an open set is finely open, namely, the fine topology is really **finer** than the original topology on E. The fine topology carries some intrinsic characteristics of the process and is usually hard to trace. The following theorem presents a lot of information on fine topology.

t:100427 **Theorem 1.3.4** (1) If f is nearly optional, then f is finely continuous if and only if $t \mapsto f(X_t)$ is right continuous. (2) If $f \in \mathbf{S}^\alpha$, f is finely continuous. (3) For $\alpha > 0$, the fine topology is generated by $\mathbf{S}^\alpha$.

Exercise 1.6 For $A \in \mathscr{E}^{no}$, A^r is finely closed and $A \cup A^r$ is the fine closure of A.

t:100428-3 **Theorem 1.3.5** For $A \in \mathscr{E}^{no}$, $X_{T_A} \in A \cup A^r$ on $\{T_A < \infty\}$ almost surely.

Proof. By definition of T_A, $\{X_{T_A} \notin A\} \subset \{T_A \circ \theta_{T_A} = 0\}$. Hence for any $x \in E$, using strong Markov property

$$\begin{aligned}
&\mathsf{P}^x(X_{T_A} \notin A \cup A^r, T_A < \infty) \\
&= \mathsf{P}^x(X_{T_A} \notin A \cup A^r, T_A \circ \theta_{T_A} = 0, T_A < \infty) \\
&= \mathsf{E}^x[\mathsf{P}^{X_{T_A}}(T_A = 0); X_{T_A} \notin A \cup A^r, T_A < \infty] = 0,
\end{aligned}$$

since $\mathsf{P}^{X_{T_A}}(T_A = 0) = 0$ for $X_{T_A} \notin A^r$. □

For an $(\mathscr{F}_t)$-stopping time T, define α-balayage kernel

$$P_T^\alpha(x, A) = \mathsf{E}^x \left[e^{-\alpha T} 1_A(X_T) \right], \ x \in E, A \in \mathscr{E}^*. \tag{1.25}$$

When $\alpha = 0$, this means $P_T(x, A) = \mathsf{P}^x(X_T \in A, T < \infty)$. If $T = T_A$, write P_T^α as P_A^α.

Lemma 1.3.6 For $g \in \mathscr{E}_+^*$,

$$P_T^\alpha U^\alpha g(x) = \mathsf{E}^x \left[\int_T^\infty e^{-\alpha t} g(X_t) dt \right]. \tag{1.26}$$

Proof. By strong Markov property,

$$\begin{aligned} P_T^\alpha U^\alpha g(x) &= \mathsf{E}^x \left[e^{-\alpha T} \mathsf{E}^{X_T} \left(\int_0^\infty e^{-\alpha t} g(X_t) dt \right) \right] \\ &= \mathsf{E}^x \left[\int_0^\infty e^{-\alpha (T+t)} f(X_{t+T}) dt \right] \\ &= \mathsf{E}^x \left[\int_T^\infty e^{-\alpha t} f(X_t) dt \right]. \end{aligned}$$

□

1:0430-2 **Lemma 1.3.7** (1) If $f \in \mathbf{S}^\alpha$, then $P_T^\alpha f \leq f$. (2) If, in addition, T is a hitting time, then $P_T^\alpha(\mathbf{S}^\alpha) \subset \mathbf{S}^\alpha$.

Proof. (1) Assume that $f(x) < \infty$. Since $t \mapsto e^{-\alpha t} f(X_t)$ is a non-negative super-martingale, we then apply the Doob's sampling theorem to get the conclusion.

(2) By Markov property, we have

$$P_t^\alpha P_T^\alpha = P_{t+T\circ\theta_t}^\alpha.$$

For T is a terminal time, $T\circ\theta_t + t \geq T$ for all $t \geq 0$ a.s. Hence if $f = U^\alpha g$, it is obvious that $P_T^\alpha U^\alpha g$ is α-super-median. T is a hitting time so $T\circ\theta_t + t \downarrow T$ as $t \downarrow 0$ and then

$$P_T^\alpha(U^\alpha \mathscr{E}_+^*) \subset \mathbf{S}^\alpha.$$

Finally the conclusion follows from Lemma 1.1.6(3) and Lemma 1.1.5(3). □

Definition 1.3.8 Let $A \in \mathscr{E}^{no}$. It is **polar** if $\mathsf{P}^x(T_A < \infty) = 0$ for all $x \in E$, **thin** if $\mathsf{P}^x(T_A > 0) = 1$ for all $x \in E$ and **semi-polar** if A is contained in a countable union of thin sets. A universally measurable subset A is **potential zero** if $U(x, A) = 0$ for all $x \in E$. The definition may apply to any subset if it is contained in a set with the respective property.

Intuitively, almost surely, X never meets a polar set and amount of time in a set of potential zero has Lebesgue measure zero. Therefore a polar set is potential zero. The following theorem asserts that semipolar sets are somewhat between.

Theorem 1.3.9 If A is a semipolar set, then almost surely $\{t : X_t \in A\}$ is at most countable.

Proof. Assume that A is thin. Let $0 < a < 1$, and $B = \{x \in A : P_A^1 1(x) \le a\}$. Set $T_1 = T_B$, $T_{n+1} = T_n + T_1 \circ \theta_{T_n}$. It is enough to show that $T_n \to \infty$ a.s. Since B is thin, $B^r =$ and $X_{T_n} \in B$ for $T_n < \infty$ by Theorem 1.3.5. By strong Markov property

$$\begin{aligned} \mathsf{E}^x[e^{-T_{n+1}}] &= \mathsf{E}^x[e^{-T_n}(e^{-T_1})\circ\theta_{T_n}] \\ &= \mathsf{E}^x[e^{-T_n}\mathsf{E}^{X_{T_n}}(e^{-T_B})] \\ &\le \mathsf{E}^x[e^{-T_n}\mathsf{E}^{X_{T_n}}(e^{-T_A})] \\ &\le a\mathsf{E}^x[e^{-T_n}], \end{aligned}$$

and hence $\mathsf{E}^x[e^{-T_n}] \to 0$, i.e., $T_n \to \infty$ a.s. □

Hence it is evident that a semipolar set is potential zero.

Theorem 1.3.10 If A is nearly optional, then $A \setminus A^r$ is semipolar.

Proof. $P_A^1 1$ is 1-excessive and finely continuous. Let

$$A_n = \{x \in A : P_A^1 1(x) \le 1 - 1/n\}.$$

Then $A \setminus A^r = \bigcup_n A_n$ and it suffices to verify that A_n is thin. For any $x \in E$, if $P_A^1 1(x) < 1$, then $P_{A_n}^1 1 < 1$ or $x \notin A_n^r$. If $P_A^1 1(x) = 1$, then x is in the finely open set $\{P_A^1 1(x) > 1 - 1/n\}$, which is disjoint with A_n, and hence $x \notin A_n^r$. This means that A_n is thin. □

Exercise 1.7 If f is α-super-median, define $\bar{f} = \lim_{t \downarrow 0} e^{-\alpha t} P_t f$. Show that $\bar{f} \in \mathbf{S}^\alpha$, $f \ge \bar{f}$ and $\{f > \bar{f}\}$ is potential zero.

Definition 1.3.11 X or (P_t) is called **transient** if U is proper, i.e., there exists a strictly positive $g \in \mathscr{E}^*$ such that $Ug < \infty$.

Since 0-potential of the semigroup $(e^{-\alpha t} P_t)$ is U^α which is proper when $\alpha > 0$, $(e^{-\alpha t} P_t)$ is always transient when $\alpha > 0$.

Lemma 1.3.12 If X is transient, then there exists strictly positive f such that $Uf \le 1$.

Proof. Let g be as in the definition. Set

$$A_n = \{g \ge \frac{1}{n}, Ug \le n\}$$

for $n \ge 1$. Then $A_n \uparrow E$. Clearly $1_{A_n} \le ng$ and A_n is contained in a finely closed set $\{Ug \le n\}$. By Theorem 1.3.5, $X_{T_{A_n}} \in \{Ug \le n\}$ for $T_{A_n} < \infty$. Now

$$\begin{aligned} U 1_{A_n}(x) &= \mathsf{E}^x \int_{T_{A_n}}^\infty 1_{A_n}(X_t) dt \\ &= P_{A_n} U 1_{A_n}(x) \\ &\le n P_{A_n} U g(x) \end{aligned}$$

$$\leq n\mathsf{E}^x[Ug(X_{T_{A_n}}), T_{A_n} < \infty] \leq n^2.$$

Then write $f = \sum_n 2^{-n}n^{-2}1_{A_n}$ which is strictly positive and $Uf(x) \leq 1$. □

It is shown in the proof that there exists $A_n \in \mathscr{E}^*$ such that $A_n \uparrow E$ and each $U1_{A_n}$ is bounded.

f:transient **Theorem 1.3.13** If X is transient and $f \in \mathbf{S}$, then there exist $g_n \in \mathscr{E}^*_+$ such that $Ug_n \uparrow f$ and both g_n and Ug_n are abounded for each n.

Proof. Take A_n as above. Set

$$h_n = n1_{A_n},\ f_n = (Uh_n) \wedge f,\ g_{n,k} = k(f_n - P_{1/k}f_n).$$

Then Uh_n is bounded, $Uh_n \uparrow +\infty$ and

$$\begin{aligned}\int_0^t P_s g_{n,k}\mathrm{d}s &= k(\int_0^t P_s f_n \mathrm{d}s - \int_{\frac{1}{k}}^{t+\frac{1}{k}} P_s f_n \mathrm{d}s)\\ &= k(\int_0^{\frac{1}{k}} P_s f_n \mathrm{d}s - \int_t^{t+\frac{1}{k}} P_s f_n \mathrm{d}s).\end{aligned}$$

Since $P_t f_n \leq P_t U_n h = \int_t^\infty P_s h_n ds \downarrow 0$ as $t \uparrow \infty$, $Ug_{n,k}$ increases with both n and k. Hence $Ug_{n,n} \uparrow f$ as $n \uparrow \infty$. □

1.4 Excessive measures

A σ-finite measure ξ is called α-**excessive measure** for X if for any $t \geq 0$, $\xi P_t^\alpha = e^{-\alpha t}\xi P_t \leq \xi$. Note that any σ-finite measure ξ on $(E, \mathscr{E})$ may be extended to a measure on $\mathscr{E}^*$ uniquely. By a result of Meyer, it follows automatically that $\xi P_t^\alpha \uparrow \xi$ as $t \downarrow 0$. The notion of excessive measures is dual to that of excessive functions. The set of α-excessive measures for X is denoted by Exc^α or $\mathrm{Exc}^\alpha(X)$ if necessary. Write Exc^0 as Exc. If ξ is a

σ-finite measure and $\xi P_t = \xi$ for all $t \geq 0$, ξ is called **invariant** for X. For example, Lebesgue measure is always invariant for Lévy processes. If μ is a measure and μU^α is σ-finite, then $\mu U^\alpha \in \mathrm{Exc}^\alpha$ which is called a **potential**.

Exercise 1.8 Let X be the translation to the right on $\mathbf{R}$ with speed one. Verify that for any $\alpha \geq 0$, $\xi(dx) = e^{-\alpha x}dx$ is invariant for $(e^{-\alpha t}P_t)$.

Lemma 1.4.1 (1) $\xi \in \mathrm{Exc}^\alpha$ if and only if $\beta \xi U^{\alpha+\beta} \leq \xi$ for any $\beta \geq 0$.
(2) If $\xi_n \in \mathrm{Exc}^\alpha$ increases to a σ-finite measure, then $\lim_n \xi_n \in \mathrm{Exc}^\alpha$.
(3) If $\xi, \eta \in \mathrm{Exc}^\alpha$, $\xi \wedge \eta \in \mathrm{Exc}^\alpha$.
(4) If $0 \leq \alpha < \beta$, then $\mathrm{Exc}^\alpha \subset \mathrm{Exc}^\beta$.

m:transient **Theorem 1.4.2** If X is transient and $\xi \in \mathrm{Exc}$, then there exist measures μ_n such that $\mu_n U \uparrow \xi$.

Proof. The proof is similar to Theorem f:transient 1.3.13. Let $A_n \uparrow E$ with $\xi(A_n) < \infty$. Fix strictly positive $g \in \mathscr{E}^*$ with $Ug \leq 1$. Set $\mu_n = n1_{A_n} \cdot \xi$. Then μ_n is finite and for any $a > 0$,

$$a\mu_n U(\{g > a\}) \leq \mu_n Ug \leq \mu_n(1) < \infty,$$

which implies $\mu_n U$ is σ-finite. Let $\eta_n = \mu_n U \wedge \xi$. We claim $\eta_n \uparrow \xi$. In fact, let $\xi_n = \int_0^1 (1_{A_n} \cdot \xi)P_t dt$. Then ξ_n increases to $\int_0^1 \xi P_t dt$ which is equivalent to ξ. Denote by f_n the density of ξ_n with respect to ξ and it follows that $\lim_n f_n > 0$ a.e. ξ. Hence the density of $(n\xi_n) \wedge \xi$ with respect to ξ is $(nf_n) \wedge 1$ which increases to 1 a.e. ξ and this implies that $\eta_n \uparrow \xi$ since

$$\xi \geq \eta_n = \int_0^\infty (n1_{A_n} \cdot \xi)P_t dt \wedge \xi = (n\xi_n) \wedge \xi.$$

Define now $\nu_n = n(\eta_n - P_{1/n}\eta_n)$ and

$$\eta_{n,k} = k\int^{1/k} \eta_n P_t dt = \int_0^1 \eta_n P_{t/k} dt.$$

Then $\nu_n U = \eta_{n,n}$. Clearly $(\eta_{n,k})$ increases with n and k since η_n is an increasing sequence of excessive measures. Hence $\nu_n U$ is increasing and $\nu_n U \leq \eta_n \leq \xi$. Moreover for each $k \geq 1$,

$$\lim \nu_n U = \lim_n \int_0^1 \eta_n P_{t/n} dt \geq \lim_n \int_0^1 \eta_k P_{t/n} dt = \eta_k.$$

Combining the fact that $\eta_n \uparrow \xi$, it leads to the conclusion. □

In 70's Meyer introduced energy functional for $\xi \in \mathrm{Exc}$ and $f \in \mathbf{S}$ which generalizes the notion of energy in classical potential theory.

Definition 1.4.3 The energy functional L on $\mathrm{Exc} \times \mathbf{S}$ is defined by

$$L(\xi, f) = \sup\{\mu(f) : \mu U \leq \xi\}. \tag{1.27}$$

We assume that X is transient when discussing energy functional for convenience. It is trivial that and $L(\mu U, f) = \mu(f)$ and if $\xi_1 \leq \xi_2$, $L(\xi_1, \cdot) \leq L(\xi_2, \cdot)$.

Lemma 1.4.4 Let $\xi \in \mathrm{Exc}$ and $f \in \mathbf{S}$. If $\mu_n U \uparrow \xi$, then $L(\mu_n U, f) \uparrow L(\xi, f)$.

Proof. It is clear that $L(\mu_n U, f)$ is increasing and we need to check $\lim_n L(\mu_n U, f) \geq L(\xi, f)$. By Theorem f:transient 1.3.13 there exist g_n such that $U g_n \uparrow f$. Hence for any $\mu U \leq \xi$,

$$\begin{aligned} \mu(f) &= \lim_n \mu(U g_n) \leq \lim_n \xi(g_n) = \lim_n \lim_k \mu_k U(g_n) \\ &= \lim_k \lim_n \mu_k U(g_n) = \lim_k \mu_k(f) = \lim_k L(\mu_k U, f). \end{aligned}$$

□

p:0501-1 **Proposition 1.4.5** Assume that $\xi \in \mathrm{Exc}$ and $f \in \mathbf{S}$.

(1) If $f_1 \leq f_2$ a.e. ξ, then $L(\xi, f_1) \leq L(\xi, f_2)$.

(2) If $f_n \uparrow f$ a.e. ξ, $L(\xi, f_n) \uparrow L(\xi, f)$.

(3) $L(\xi, Uf) = \xi(f)$.

(4) If $\xi_n \uparrow \xi$, then $L(\xi_n, f) \uparrow L(\xi, f)$.

(5) If $L(\xi, f) = 0$, then $f = 0$ a.e. ξ.

Proof. (1) The set $A = \{f_1 > f_2\}$ is finely open and ξ-null. For any $\mu U \leq \xi$, $\mu U(A) = 0$ implies that $\mu(A) = 0$, or $f_1 \leq f_2$ a.e. μ. Hence $\mu(f_1) \leq \mu(f_2)$, i.e., $L(\xi, f_1) \leq L(\xi, f_2)$. (2) If $f_n \uparrow f$, then take $\mu_n U \uparrow \xi$ and we have

$$\begin{aligned}\lim_n L(\xi, f_n) &= \lim_n \lim_k L(\mu_k U, f_n) = \lim_k \lim_n \mu_k(f_n) \\ &= \lim_k \mu_k(f) = \lim_k L(\mu_k U, f) = L(\xi, f).\end{aligned}$$

(3) Take $\mu_n U \uparrow \xi$. Then $L(\xi, Uf) = \lim_n \mu_n(Uf) = \lim_n (\mu_n U)f = \xi(f)$. (4) It suffices to show that $\lim_n L(\xi_n, f) \geq L(\xi, f)$. Take any $\mu U \leq \xi$ and $Ug_n \uparrow f$. Then

$$\begin{aligned}\mu(f) &= \lim_n \mu(Ug_n) \leq \lim_n \xi(g_n) \\ &= \lim_n \lim_k \xi_k(g_n) = \lim_n \lim_k L(\xi_k, Ug_n) \\ &= \lim_k L(\xi_k, f).\end{aligned}$$

(5) Take $Ug_n \uparrow f$. By (3), $\xi(g_n) = 0$. Since $\alpha\xi U^\alpha \uparrow \xi$, $\xi U(g_n) = \lim_{\alpha\downarrow 0} \xi U^\alpha(g_n) = 0$ and hence $\xi(f) = 0$. □

c:0501-1 **Corollary 1.4.6** Let $\xi \in \text{Exc}$ and $f \in \mathbf{S}$. (1) If ξ is purely excessive, i.e., $\xi P_t \downarrow 0$ as $t \uparrow \infty$, then

$$L(\xi, f) = \lim_{t\downarrow 0} t^{-1}\langle \xi - \xi P_t, f\rangle = \lim_{\alpha\uparrow+\infty} \alpha\langle \xi - \alpha\xi U^\alpha, f\rangle. \tag{1.28}$$

. (2) If $f < \infty$ and $P_t f \downarrow 0$ as $t \uparrow +\infty$,

$$L(\xi, f) = \lim_{t\downarrow 0} t^{-1}\langle \xi, f - P_t f\rangle = \lim_{\alpha\uparrow\infty} \alpha\langle \xi, f - \alpha U^\alpha f\rangle. \tag{1.29}$$

Proof. For (1), define $\mu_t = t^{-1}(\xi - \xi P_t)$. Then it is routine to check that $\mu_t U \uparrow \xi$ as $t \downarrow 0$ and the conclusion follows from Proposition p:0501-1 1.4.5(4). The other statements are proved similarly. □

It is seen from above that formally

$$L(\xi, f) = \langle -\xi L, f \rangle = \langle \xi, -Lf \rangle$$

where L, on the right, is the infinitesimal generator of (P_t), which is exactly what the energy means classically.

Given a σ-finite measure m on $(E, \mathscr{E}^*)$. Define P^m as

$$\mathsf{P}^m(H) = \int_E \mathsf{P}^x(H)m(dx), \; H \in \mathscr{F}_\infty, \tag{1.30}$$

The process X still has strong Markov property with respect to measure P^m: for any stopping time T and non-negative random variable Y on $(\Omega, \mathscr{F}_\infty)$, it holds P^m-a.s., on $\{T < \infty\}$,

$$\mathsf{E}^m[Y \circ \theta_T] = \mathsf{E}^{X_T}[Y]. \tag{1.31}$$

Fix $m \in \text{Exc}$. A nearly optional set A is called m-polar if $\mathsf{P}^m(T_A < \infty) = 0$, or $\mathsf{P}^x(T_A < \infty) = 0$ for m-almost all $x \in E$. A polar set is certainly m-polar. In general, we may define the **capacity** of A by

$$\Gamma(A) = L(m, P_A 1). \tag{1.32}$$

It is clear that A is m-polar if and only if $\Gamma(A) = 0$. Moreover we have the following properties.

Proposition 1.4.7 Let $A, B, B_n \in \mathscr{E}^{no}$.
(1) If $A \subset B$, then $\Gamma(A) \leq \Gamma(B)$.
(2) If $B_n \uparrow B$, then $\Gamma(B_n) \uparrow \Gamma(B)$.
(3) $\Gamma(A \cup B) + \Gamma(A \cap B) \leq \Gamma(A) + \Gamma(B)$.

Lemma 1.4.8 Let $A \in \mathscr{E}^{no}$ be finely open. If $m(A) = 0$, then A is m-polar.

Proof. Since $m \in \text{Exc}$, $m(A) = 0$ implies that $mU(A) = 0$, i.e., P^m-a.e., the amount of time that X stays in A has Lebesgue measure zero and it can be seen that X could never meet A, because, roughly speaking, when A is finely open and X is right continuous, a sample path of X meeting A would stay in A for a time interval.

A rigorous proof goes this way. Let $g(x) = 1 - \mathsf{E}^x(e^{-T_{A^c}})$, which is zero on A^c and strictly positive on A. Then g is finely continuous and $t \mapsto g(X_t)$ is right continuous. If $T_A < \infty$, then there exists $t > 0$ such that $g(X_t) > 0$. The right continuity implies that $g(X_s) > 0$ for $s \in [t, t+\delta)$ with some $\delta > 0$ and hence $\int_0^\infty g(X_t)dt > 0$. However $\mathsf{E}^m \int_0^\infty g(X_t)dt \le mU(A) = 0$ and therefore $\mathsf{P}^m(T_A < \infty) = 0$. □

1.5 Additive functionals and Revuz measures

Additive functionals play an important role in theory of Markov processes. If $B = (B_t)$ is a standard Brownian motion on $\mathbf{R}^d$ and ϕ is a bounded Borel function on $\mathbf{R}^d$, it is well-known that the function $u = u(t, x)$ satisfying Schrödinger equation with initial condition

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u - \phi u, \\ u(0, x) &= f(x) \end{aligned} \tag{1.33}$$

may be written as

$$u(t, x) = \mathsf{E}^x\left(f(B_t)\exp\left(-\int_0^t \phi(B_s)ds\right)\right), \tag{1.34}$$

which is called the Feynman-Kac formula. What we interested here is the path integral

$$\int_0^t \phi(B_s)ds$$

which has additivity

$$\begin{aligned}\left(\int_0^s \phi(B_u)du\right)\circ\theta_t &= \int_0^s \phi(B_u\circ\theta_t)du \\ &= \int_0^s \phi(B_{u+t})du = \int_t^{s+t} \phi(B_u)du \\ &= \int_0^{t+s} \phi(B_u)du - \int_0^t \phi(B_u)du\end{aligned}$$

for all $t, s \geq 0$ a.s.

Now we give the definition of additive functionals. Let

$$X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, \mathsf{P}^x)$$

be a right process on E with transition semigroup (P_t).

Definition 1.5.1 A non-negative right continuous process $A = (A_t)$ is called a **raw additive functional**, or simply RAF, of X if almost surely

$$A_{t+s} = A_t + A_s\circ\theta_t$$

holds for all $t, s \geq 0$. A raw additive functional is an additive functional, or simply AF, if it is adapted. A continuous additive functional of X is called a PCAF simply.

By way of perfection, the additivity may be weakened: for any $t, s \geq 0$, it holds almost surely

$$A_{t+s} = A_t + A_s\circ\theta_t.$$

Clearly a raw additive functional A is always increasing, $A_0 = 0$ and so we denote by dA_t the measure induced by $t \mapsto A_t$. For a non-negative measurable function f on E,

$$t \mapsto \int_0^t f(X_s)dA_s$$

is still a raw additive functional of X and denoted by $f*A$. If A is an additive functional, so is $f*A$.

Lemma 1.5.2 Let $m \in \text{Exc}$ and A a raw additive functional of X. Define

$$\varphi(t) = \mathsf{E}^m(A_t),\ t \geq 0.$$

Then $\varphi(0) = 0$, φ is increasing and concave. Furthermore $t \mapsto \frac{\varphi(t)}{t}$ decreases and

$$\lim_{t\downarrow 0} \frac{\varphi(t)}{t} = \lim_{\beta\uparrow\infty} \beta\mathsf{E}^m \int_0^\infty e^{-\beta t}dA_t. \tag{1.35}$$

Proof. It is obvious that $\varphi(0) = 0$ and φ is increasing. Then φ has right and left limits at any point. By Markov property, for $t, u \geq 0$

$$\varphi(t+u) = \mathsf{E}^m(A_{t+u}) = \varphi(t) + \mathsf{E}^m(A_u \circ \theta_t) = \varphi(t) + \mathsf{E}^{mP_t}(A_u)$$

and then $\varphi(t+u) \leq \varphi(t) + \varphi(u)$. It follows that if φ is infinite at some point $t_0 > 0$ then φ is infinite identically on $(0,\infty)$ since $n\varphi(t_0/n) \geq \varphi(t_0)$ and φ is increasing. We may assume that ϕ is finite on $[0,\infty)$. In this case we have

$$\varphi(t+u) - \varphi(t) \leq \varphi(s+u) - \varphi(s) \tag{1.36}$$

for $t > s \geq 0$ and $u \geq 0$. Indeed, by Markov property, we have

$$\begin{aligned}
\varphi(t+u) - \varphi(t) &= \mathsf{E}^m(A_{t+u} - A_t) \\
&= \mathsf{E}^m\left[(A_{s+u} - A_s)\circ\theta_{t-s}\right] \\
&= \mathsf{E}^{mP_{t-s}}(A_{s+u} - A_s) \\
&\leq \mathsf{E}^m(A_{s+u} - A_s) = \varphi(s+u) - \varphi(s).
\end{aligned}$$

An easy consequence is

$$\varphi(\frac{t+s}{2}) \geq \frac{\varphi(t)+\varphi(s)}{2} \tag{1.37}$$

for $t, s \geq 0$. For $t > 0$ and $\varepsilon > 0$, $2\varphi(t-\varepsilon) \geq \varphi(z+\varepsilon) + \varphi(z-3\varepsilon)$ and hence $\varphi(t-) \geq \varphi(t+)$. This implies that φ is continuous. Therefore φ is concave and $t \mapsto \frac{\varphi(t)}{t}$ decreases.

It is easy to verify that φ is Lipshitz continuous on $(\delta, +\infty)$ for any $\delta > 0$. Let φ' be the right hand derivative of φ. Then it is right continuous and decreasing. Now we have

$$\begin{aligned}
\beta \mathsf{E}^m \int_0^\infty e^{-\beta t} dA_t &= \beta \int_0^\infty e^{-\beta t} d\varphi(t) \\
&= \int_0^\infty \beta e^{-\beta t} \varphi'(t) dt \\
&= \int_0^\infty e^{-t} \varphi'(t/\beta) dt
\end{aligned}$$

and hence $\lim_{\beta\uparrow\infty} \beta \mathsf{E}^m \int_0^\infty e^{-\beta t} dA_t \uparrow \varphi'(0)$. □

Applying this lemma, for any non-negative measurable function f on E,

$$\frac{1}{t}\mathsf{E}^m \left(\int_0^t f(X_s) dA_s \right)$$

is increasing as t decreases. Write its limit as $t \downarrow 0$ as $\rho_A^m(f)$, i.e.,

$$\rho_A^m(f) = \lim_{t\downarrow 0} \frac{1}{t}\mathsf{E}^m \int_0^t f(X_s) dA_s = \sup_{t>0} \frac{1}{t}\mathsf{E}^m \int_0^t f(X_s) dA_s, \tag{1.38}$$

which indicates its dependence on A, m and also X of course, though it is not shown on the notation. It can be seen that ρ_A^m is a measure on E and called the Revuz measure (or characteristic measure) of A with respect to m computed against X, or simply Revuz measure of A if no confusion will be caused. Obviously a Revuz measure will not charge any m-polar set and the Revuz measure of a PCAF will not charge any m-semipolar set.

Clearly $A_t = t$ is a trivial additive functional for any process. Then

$$\mathsf{E}^m \int_0^t f(X_s)ds = \int_0^t mP_s(f)ds.$$

Since $mP_s \uparrow m$ as $s \downarrow 0$,

$$\frac{1}{t}\mathsf{E}^m \int_0^t f(X_s)ds \uparrow m(f)$$

as $t \downarrow 0$, i.e., its Revuz measure is m itself. Another easy consequence is that for a non-negative measurable $g \geq 0$ on E,

$$\rho^m_{g*A} = g \cdot \rho^m_A. \tag{1.39}$$

Introduce α-potential of A by

$$U^\alpha_A f(x) = \mathsf{E}^x \int_0^\infty e^{-\alpha t} f(X_t)dA_t,\ \alpha \geq 0, f \in \mathscr{E}^*_+. \tag{1.40}$$

We have similarly

$$U^\gamma_A = U^\alpha_A + (\alpha - \gamma)U^\gamma U^\alpha_A \tag{1.41}$$ e:resolvent2

for $\alpha > \gamma \geq 0$. If A is a raw additive functional of X, we have

$$\begin{aligned} P^\alpha_t U^\alpha_A f(x) &= e^{-\alpha t}\mathsf{E}^x \left[\left(\int_0^\infty e^{-\alpha u} f(X_u)dA_u \right) \circ \theta_t \right] \\ &= \mathsf{E}^x \left(\int_t^\infty e^{-\alpha u} f(X_u)dA_u \right) \end{aligned}$$

and it can be seen that $U^\alpha_A f \in \mathbf{S}^\alpha$.

l:0516-1 **Lemma 1.5.3** If $\mu U \in$ Exc and A is a raw additive functional, then

$$\rho^{\mu U}_A = \mu U_A.$$

Proof. Fix a measurable function $f \geq 0$ on E and we have by Fubini's theorem

$$\begin{aligned}
\rho_A^{\mu U}(f) &= \lim_{t\downarrow 0} \frac{1}{t} \mathsf{E}^{\mu U} \int_0^t f(X_s) dA_s \\
&= \lim_{t\downarrow 0} \frac{1}{t} \int \mu(dx) \int U(x,dy) \mathsf{E}^y \int_0^t f(X_s) dA_s \\
&= \lim_{t\downarrow 0} \frac{1}{t} \mathsf{E}^{\mu} \int_0^\infty \left(\int_0^t f(X_s) dA_s \right) \circ \theta_u du \\
&= \lim_{t\downarrow 0} \frac{1}{t} \mathsf{E}^{\mu} \int_0^\infty du \int_u^{t+u} f(X_s) dA_s \\
&= \lim_{t\downarrow 0} \frac{1}{t} \mathsf{E}^{\mu} \int_0^\infty (s - (s-t)^+) f(X_s) dA_s.
\end{aligned}$$

Then the monotone convergence theorem gives the conclusion. □

Theorem 1.5.4 If X is transient,

$$\rho_A^m(f) = L(m, U_A f). \tag{1.42}$$

Proof. Since X is transient, there exists a sequence of potentials $\{\mu_n U\}$ which increases to m. By Lemma 1.5.3 and Proposition 1.4.5(4), we have

$$\rho_A^m(f) = \lim_n \rho_A^{\mu_n U}(f) = \lim_n \mu_n U_A f = \lim_n L(\mu_n U, U_A f) = L(m, U_A f).$$

That completes the proof. □

For $\alpha > 0$, the process corresponding the transition semigroup (P_t^α) is called α-subprocess and denoted by X^α, which is always transient. A raw additive functional A of X is also a raw additive functional of X^α. We may compute the Revuz measure of A with respect to $m \in \text{Exc} \subset \text{Exc}^\alpha$ against X^α and denote it by $\rho_A^{m,\alpha}$.

Exercise 1.9 If A is a RAF of X, then A is also a RAF of X^α and

$$\rho_A^{m,\alpha}(f) = \lim_{t\downarrow 0}\frac{1}{t}\mathsf{E}^m\int_0^t e^{-\alpha s}f(X_s)\mathrm{d}A_s.$$

Therefore $\rho_A^m = \uparrow \lim_{\alpha\downarrow 0}\rho_A^{m,\alpha}$.

It is then obvious that

$$\rho_A^{m,\alpha}(f) = L^\alpha(m, U_A^\alpha f), \tag{1.43}$$

where L^α is the energy functional of X^α.

Lemma 1.5.5 The Revuz measure $\rho_A^{m,\alpha}$ does not depend on $\alpha > 0$ and therefore $\rho_A^{m,\alpha} = \rho_A^m$.

Proof. Let $\alpha > \gamma > 0$. Then by the resolvent equation, we have

$$\begin{aligned} L^\alpha(m, U^\alpha f) = m(f) &= L^\gamma(m, U^\gamma f) \\ &= L^\gamma(m, U^\alpha f + (\alpha-\gamma)U^\gamma U^\alpha f) \end{aligned}$$

for non-negative and measurable f on E. If $h \in \mathbf{S}^\alpha$, there exists a sequence of potentials $U^\alpha f_n$ increasing to h and hence it follows that $h+(\alpha-\gamma)U^\gamma h \in \mathbf{S}^\gamma$ and

$$L^\alpha(m,h) = L^\gamma(m, h+(\alpha-\gamma)U^\gamma h). \tag{1.44}$$ e:energy2

Then by the resolvent equation (1.41) and plugging $U_A^\alpha f \in \mathbf{S}^\alpha$ in (1.44), we have the conclusion. □

1.6 Dual processes

Suppose that $X = (X_t, \mathsf{P}^x)$ and $\widehat{X} = (\widehat{X}, \widehat{\mathsf{P}}^x)$ are two right Markov processes on E with transition semigroups (P_t) and $(\widehat{P}_t)$ respectively. Let m be a σ-finite measure on E. We may assume that both are defined on the canonical space of right continuous paths on E.

Definition 1.6.1 The processes X and $\widehat{X}$ are called **dual** relative to m if

$$\int g(x)P_tf(x)m(dx) = \int f(x)\widehat{P}_tg(x)m(dx), \tag{1.45}$$

e:duality

for non-negative measurable functions f, g on $(E, \mathscr{E}^*)$.

Clearly in this case m must be excessive for both X and $\widehat{X}$ and if $\widehat{h} \in \mathbf{S}(\widehat{X})$, then $\widehat{h}\cdot m \in \text{Exc}(X)$. Naturally we have measures P^m and $\widehat{\mathsf{P}}^m$:

$$\mathsf{P}^m = \int_E \mathsf{P}^x m(\mathrm{d}x),\ \widehat{\mathsf{P}}^m = \int_E \widehat{\mathsf{P}}^x m(dfx). \tag{1.46}$$

Define the reversal operator γ_t on $\{t < \zeta\}$:

$$\gamma_t\omega(s) = \omega(t-s),\ s \in [0,t],$$

i.e., $X_s\circ\gamma_t = X_{t-s}$.

l:reverse **Lemma 1.6.2** If Y is a $\mathscr{F}_t^0$-measurable non-negative random variable, then

$$\mathsf{E}^m(Y\circ\gamma_t; t<\zeta) = \widehat{\mathsf{E}}^m(Y; t<\zeta). \tag{1.47}$$

Proof. By the monotone class theorem, it suffices to verify for

$$Y = f_1(X_{t_1})\cdots f_n(X_{t_n}),$$

where $0 = t_1 < \cdots < t_n = t$. The equation (1.45) is equivalent to $m(\mathrm{d}x)P_t(x,\mathrm{d}y) = m(\mathrm{d}y)\widehat{P}_t(y,\mathrm{d}x)$. Hence

$$\begin{aligned}
\mathsf{E}^m(Y\circ\gamma_t) &= \mathsf{E}^m[f_n(X_0)f_{n-1}(X_{t_n-t_{n-1}})\cdots f_2(X_{t_n-t_2})f_1(X_t)] \\
&= \int f_n(x_n)\cdots f_1(x_1)m(\mathrm{d}x_n)P_{t_n-t_{n-1}}(x_n,\mathrm{d}x_{n-1})\cdots P_{t_2}(x_2,\mathrm{d}x_1) \\
&= \int f_n(x_n)\cdots f_1(x_1)\widehat{P}_{t_n-t_{n-1}}(x_{n-1},\mathrm{d}x_n)\cdots \widehat{P}_{t_2}(x_1,\mathrm{d}x_2)m(\mathrm{d}x_1) \\
&= \widehat{\mathsf{E}}^m(f_1(X_{t_1})f_2(X_{t_2})\cdots f_n(X_{t_n})) \\
&= \widehat{\mathsf{E}}^m(Y).
\end{aligned}$$

□

The following theorem is due to J.B.Walsh [**?**].

Theorem 1.6.3 P^m-a.e. $X = (X_t)$ has left limits for $t \in (0, \zeta)$. If $f \in \mathbf{S}(\widehat{X}^\alpha)$, then P^m-a.s., $t \mapsto f(X_{t-})$ is left continuous.

Proof. Let $X|_Q$ denote the process $\{X_t : t \geq 0 \text{ rational}\}$. Fix a rational $t > 0$. The event

$$A_t = \{X|_Q \text{ has left limits at all} s \in (0,t), t < \zeta\}$$

is measurable with respect to $\mathscr{F}_t^0$. Clearly if $t < \zeta$,

$$\gamma_t^{-1}(A_t) = \{X|_Q \text{ has right limits at all} s \in (0,t), t < \zeta\}. \tag{1.48}$$

1:reverse

By Lemma 1.6.2, $\mathsf{P}^m(A_t^c) = \widehat{\mathsf{P}}^m(\gamma^{-1}A_t^c)$ and it follows that

$$\mathsf{P}^m\left(\bigcup_{t \in Q} A_t^c\right) = \widehat{\mathsf{P}}^m\left(\bigcup_{t \in Q} \gamma_t^{-1}A_t^c\right) = 0.$$

This means that P^m-a.e., $X|_Q$ has left limits for $t < \zeta$ and, due to the right continuity of X, X has left limits for $t < \zeta$. The second assertion will be proved later. □

Moreover $(\widehat{U}f) \cdot m = (f \cdot m)U$ by duality for measurable $f \geq 0$. Note that X and $\widehat{X}$ are mutually dual and so the dual statement of any assertion also holds. Define the energy functional of X on $\mathbf{S}(\widehat{X}) \times \mathbf{S}(X)$

$$L(\widehat{h}, h) = L(\widehat{h} \cdot m, h),\ \widehat{h} \in \mathbf{S}(\widehat{X}), h \in \mathbf{S}(X), \tag{1.49}$$

and $\widehat{L}$ the energy functional of $\widehat{X}$, and L^α for dual X^α and $\widehat{X}^\alpha$ similarly. Note that we use the same L for two kinds of energy functionals.

Lemma 1.6.4 If X is transient, then

$$L^\alpha(\widehat{h}, h) = \widehat{L}^\alpha(h, \widehat{h}) \tag{1.50}$$

for $h \in \mathbf{S}(X)$ and $\widehat{h} \in \mathbf{S}(\widehat{X})$, *Proof.* Let $h = Uf$ with measurable $f \geq 0$. Then we have

$$\begin{aligned} L(\widehat{h}, h) &= \int f(x)\widehat{h}(x)m(dx) \\ &= \widehat{L}((f \cdot m)\widehat{U}, \widehat{h}) \\ &= \widehat{L}(Uf \cdot m, \widehat{h}). \end{aligned}$$

Now for any $h \in \mathbf{S}(X)$, we may take a sequence Uf_n of potentials increasing to h and it holds

$$L(\widehat{h}, Uf_n) = \widehat{L}(Uf_n, \widehat{h}).$$

Let n go to infinity and applying Proposition 1.4.5(2)(4) we get to the conclusion. □

We now prove the useful Revuz formula.

Theorem 1.6.5 Let A be a RAF of X and $\widehat{A}$ a RAF of $\widehat{X}$. Then

$$\int_E f(x)U_A^\alpha g(x)\rho_{\widehat{A}}^m(dx) = \int_E g(x)\widehat{U}_{\widehat{A}}^\alpha f(x)\rho_A^m(dx). \tag{1.51}$$

Chapter 2

Potential analysis of multiplicative functionals

2.1 Multiplicative functionals

Definition 2.1.1 A right continuous adapted stochastic process $M = (M_t)$ valued in $[0, 1]$ on Ω is called a multiplicative functional of X if almost surely

$$M_{t+s} = M_t \cdot M_s \circ \theta_t, \ t, s \geq 0. \tag{2.1}$$

Two trivial examples of multiplicative functionals are

$$M_t = \exp\left(-\int_0^t f(X_s)ds\right), \ f \geq 0;$$
$$M_t = 1_{\{t<T_A\}},$$

where T_A is a hitting time. Actually if T is a terminal and stopping time, then $M_t = 1_{\{t<T\}}$ is a mutiplicative functional. On the other hands, if A is an AF, then its usual exponential $M_t = e^{-A_t}$ is a multiplicative functional.

Given a multiplicative functional M. Clearly M is decreasing and $M_0 = M_0^2$, i.e., $M_0 = 0$ or 1 almost surely. If $M_0 = 0$, M is identically 0. Let

$$E_M = \{x \in E : \mathsf{P}^x(M_0 = 1) = 1\},$$
$$S_M = \inf\{t : M_t = 0\}.$$

The set E_M is called the set permanent points for M and S_M the life time of M. If $S_M \geq \zeta$, then we say M never vanishes.

Chapter 3

Dirichlet forms

In this part we study the analytic aspect of Markov semigroups and their associated Markov processes. We outline the beautiful theory of Markov semigroups, which is the natural product by combining Hille-Yosida's theory of one-parameter semigroups with the Markov property.

We begin with a short summary of Hille-Yosida's theory of semigroups on Banach spaces, which is necessary for the study of Markov semigroups. We then present a few special features about symmetric Markov semigroups and their associated Dirichlet spaces.

3.1 Contraction semigroups and infinitesimal generators

Recall that a time-homogenous Markov chain $(X_t)_{t\geq 0}$ on a discrete state space M is described through its transition probability

$$p_{ij}(t) = P\left(X_t = j | X_0 = i\right).$$

The transition matrix $P(t) = (p_{ij}(t))$ satisfies the Chapman-Kolmogorov equation

$$p_{ij}(s+t) = \sum_{k \in M} p_{ik}(s) p_{kj}(t) \qquad \text{for any} \quad s,\, t > 0 \ .$$

The transition matrix $(p_{ij}(t))$ allows us to define a linear operator P_t on the space $C_b(M)$ of bounded (continuous) functions on M by

$$(P_t f)(i) = \sum_{j \in M} f(j) p_{ij}(t) \qquad \forall i \in M \ .$$

If $C_b(M)$ is endowed with the supremum norm

$$\|f\| = \sup_{i \in M} |f(i)| \qquad \forall f \in C_b(M)$$

then $C_b(M)$ is a Banach space, and each linear operator $P_t : C_b(M) \to C_b(M)$ is a contraction $\|P_t f\| \le \|f\|$ for any $f \in C_b(M)$. The Chapman-Kolmogorov equation implies that $(P_t)_{t>0}$ is a *semigroup* on $C_b(M)$, i.e. $P_t(P_s f) = P_{t+s} f$. It is thus not surprising that the theory of 1-parameter semigroups plays a fundamental rôle in the theory of Markov processes.

Let now B be a (real or complex) Banach space with a norm $\|\cdot\|$. Typical examples are L^p-spaces on σ-finite measure spaces. A linear operator $T : B \to B$ is bounded if there is a non-negative constant C such that $\|T(x)\| \le C\|x\|$ for any $x \in B$. In this case, the least $C \ge 0$ such that the previous statement is true is called the norm of T, denoted by $\|T\|$. A basic fact in functional analysis is that a linear operator is continuous if and only if it is bounded. For simplicity, if no confusion may arises, $T(x)$ will be simply written as Tx.

A linear operator $T : B \to B$ is called a contraction if $\|T\| \le 1$.

A one-parameter family $(P_t)_{t \ge 0}$ of bounded linear operators $P_t : B \to B$ is a semigroup (of linear operators) on B, if

1. $P_0 = I$ the identity operator on B,
2. $(P_t)_{t\geq 0}$ satisfies the semigroup property: $P_{t+s} = P_t P_s$ for every s, $t \geq 0$, where $P_t P_s$ is the composition of operators, that is, $P_t P_s(f) = P_t(P_s(f))$.

A semigroup $(P_t)_{t\geq 0}$ is *strongly continuous* if

$$\lim_{t\downarrow 0} P_t x = x \quad \text{for every } x \in B \ .$$

A semigroup $(P_t)_{t\geq 0}$ is a semigroup of contractions (or called *contraction semigroup*) if each P_t is a contraction on B, that is, $\|P_t\| \leq 1$.

If $(P_t)_{t\geq 0}$ is a semigroup on B, then its infinitesimal generator (or simply generator) is the (unbounded) linear operator $(L, D(L))$ defined by

$$Lx = \lim_{t\downarrow 0} \frac{1}{t} \left(P_t x - x\right)$$

where $x \in D(L)$, and

$$D(L) = \left\{ x \in B : \lim_{t\downarrow 0} \frac{1}{t} \left(P_t x - x\right) \text{ exists} \right\} .$$

The following theorem summarizes the basic properties of the infinitesimal generator of a semigroup, the proofs leave for the reader as an exercise.

th1.1 **Theorem 3.1.1** Let $(P_t)_{t\geq 0}$ be a strongly continuous semigroup on Banach space B, and let L be its infinitesimal generator with domain $D(L)$.

1) The map $t \to P_t x$ is uniformly continuous for every $x \in B$, and

$$\lim_{h\downarrow 0} \frac{1}{h} \int_t^{t+h} (P_s x) \, \mathrm{d}s = P_t x \quad \forall t \geq 0, \;\; x \in B. \tag{3.1}$$ se001

Moreover, $\int_0^t (P_s x) \mathrm{d}s \in D(L)$ for $t > 0$, $x \in B$, and

$$L \int_0^t (P_s x) \, \mathrm{d}s = P_t x - x \ .$$

2) If $x \in D(L)$, so is $P_t x$,

$$\frac{d}{dt} P_t x = L(P_t x) = P_t(Lx) \tag{3.2}$$ sse04

and

$$\begin{aligned} P_t x - P_s x &= \int_s^t (P_u L x)\,\mathrm{d}u \\ &= \int_s^t (L P_u x)\,\mathrm{d}u = L\left(\int_s^t (P_u x)\,\mathrm{d}u\right) . \end{aligned}$$

As a consequence, $D(L)$ is dense in B. Therefore the infinitesimal generator L of a strongly continuous semigroup on a Banach space B is densely defined linear operator on B, each P_t leaves $D(L)$ invariant, L commutates with P_t, and for any $x \in B$ the integral $\int_0^t (P_s x)\mathrm{d}s$ (for $t > 0$) is an element in $D(L)$.

Generally, the infinitesimal generator L of a strongly continuous semigroup on a Banach space B is unbounded, and in general $D(L) \neq B$. To further investigate the properties of the densely defined linear operator L, we need the following

def01 **Definition 3.1.2** The graph of a densely defined linear operator $(T, D(T))$ on a Banach space B is

$$G(T) = \{(x, Tx) : x \in D(T)\}$$

which is a subset of the product space $B \times B$ (endowed with norm $\|(x, y)\| = \|x\| + \|y\|$). T is called a *closed operator* if $G(T)$ is a closed subset of $B \times B$ (and thus $G(T)$ itself is a Banach space). In other words, T is closed if for every sequence $\{x_n\}$ of $D(T)$ such that $x_n \to x$ and $Tx_n \to y$, then $x \in D(T)$ and $Tx = y$.

propy01 **Proposition 3.1.3** The infinitesimal generator $(L, D(L))$ of a strongly continuous contraction semigroup $(P_t)_{t \geq 0}$ is a closed operator.

Proof. Suppose $x_n \in D(L) \to x$ and $Lx_n \to y$. We have to show that $x \in D(L)$ and $Lx = y$. Since $Lx_n \to y$, $\{Lx_n\}$ is bounded in B and

$$\|P_s L(x_n)\| \leq \|L(x_n)\| \leq \sup \|L(x_n)\| \qquad \text{for all } s \geq 0.$$

A computation leads us to

$$\begin{aligned}
\frac{P_t x - x}{t} &= \frac{1}{t}\left(P_t(\lim_{n\to\infty} x_n) - \lim_{n\to\infty} x_n\right) \\
&= \lim_{n\to\infty} \frac{1}{t}\left(P_t(x_n) - x_n\right) && \text{(as } P_t \text{ is continuous)} \\
&= \lim_{n\to\infty} \frac{1}{t}\int_0^t P_s L(x_n)\mathrm{d}s \\
&= \frac{1}{t}\int_0^t \lim_{n\to\infty} P_s L(x_n)\mathrm{d}s && \text{(Dominated Convergence)} \\
&= \frac{1}{t}\int_0^t P_s y \mathrm{d}s
\end{aligned}$$

and

$$\lim_{t\downarrow 0} \frac{1}{t}\left(P_t x - x\right) = \lim_{t\downarrow 0} \frac{1}{t}\int_0^t P_s y \mathrm{d}s = y\ .$$

Therefore $x \in D(L)$ and $Lx = y$. □

It is usually very difficult to determine the domain $D(L)$ of the infinitesimal generator L. Since the graph $G(L)$ of L obviously determines L uniquely, and since $G(L)$ is a closed subspace of $B \times B$, any dense subset of $G(L)$ will determine $G(L)$ and therefore the closed linear operator L uniquely. A subset C of $D(L)$ is a *core* for a closed linear operator $(L, D(L))$ if $\{(x, Lx) : x \in C\}$ is dense in $G(L)$. Precisely, for any $x \in D(L)$ there is a sequence $\{x_n\}$ in C such that $x_n \to x$ and $Lx_n \to y$ for some $y \in B$.

Another important concept associated with a strongly continuous contraction semigroup $(P_t)_{t\geq 0}$ is the *resolvent* $\{R_\lambda : \lambda > 0\}$ which we have met in the previous chapter. By definition

$$R_\lambda = \int_0^\infty e^{-\lambda t} P_t \mathrm{d}t\ .$$

Since for every $x \in B$

$$\begin{aligned} \|R_\lambda x\| &= \left\| \int_0^\infty e^{-\lambda t} (P_t x) \, \mathrm{d}t \right\| \\ &\leq \frac{1}{\lambda} \|x\| , \end{aligned}$$

each R_λ (for $\lambda > 0$) is a bounded linear operator of B with $\|R_\lambda\| \leq 1/\lambda$. $\{R_\lambda : \lambda > 0\}$ is a commutative family of bounded linear operators of B, and the semigroup property of $(P_t)_{t \geq 0}$ implies that $\{R_\lambda : \lambda > 0\}$ satisfies the resolvent equation:

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu \quad \forall \lambda, \mu > 0 \ .$$

In particular, the region $\{R_\lambda x : x \in B\}$ does not depend on $\lambda > 0$.

propy03 **Proposition 3.1.4** If $(L, D(L))$ is the infinitesimal generator of a strongly continuous semigroup $(P_t)_{t \geq 0}$ of contractions of B, then for any $\lambda > 0$, $\lambda - L$ (where λ means the multiplier λI) is invertible and $R_\lambda = (\lambda - L)^{-1}$. In particular, for every $\lambda > 0$ the region of R_λ

$$\{R_\lambda x : x \in B\} \subset D(L) \ .$$

Proof. We only need to show that for every $x \in B$

$$(\lambda - L)(R_\lambda x) = x \ .$$

Firstly show that $R_\lambda x$. In fact

$$\begin{aligned} P_h (R_\lambda x) - R_\lambda x &= P_h \int_0^\infty \mathrm{e}^{-\lambda t} (P_t x) \, \mathrm{d}t - \int_0^\infty \mathrm{e}^{-\lambda t} (P_t x) \, \mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-\lambda t} (P_{t+h} x) \, \mathrm{d}t - \int_0^\infty \mathrm{e}^{-\lambda t} (P_t x) \, \mathrm{d}t \\ &= \mathrm{e}^{\lambda h} \int_h^\infty \mathrm{e}^{-\lambda t} (P_t x) \, \mathrm{d}t - \int_0^\infty \mathrm{e}^{-\lambda t} (P_t x) \, \mathrm{d}t \end{aligned}$$

$$= \left(\mathrm{e}^{\lambda h} - 1\right) \int_h^\infty \mathrm{e}^{-\lambda t} \left(P_t x\right) \mathrm{d}t - \int_0^h \mathrm{e}^{-\lambda t} \left(P_t x\right) \mathrm{d}t$$

and we then have

$$\begin{aligned} \lim_{h \downarrow 0} \frac{1}{h} \left(P_h \left(R_\lambda x\right) - R_\lambda x\right) &= \lim_{h \downarrow 0} \frac{\mathrm{e}^{\lambda h} - 1}{h} \int_h^\infty \mathrm{e}^{-\lambda t} \left(P_t x\right) \mathrm{d}t \\ &- \lim_{h \downarrow 0} \frac{1}{h} \int_0^h \mathrm{e}^{-\lambda t} \left(P_t x\right) \mathrm{d}t \\ &= \lambda R_\lambda x - x \ . \end{aligned}$$

Therefore $R_\lambda x \in D(L)$ and

$$L\left(R_\lambda x\right) = \lambda R_\lambda x - x$$

which proves the claim. □

Therefore any real $\lambda > 0$ belongs to the resolvent set of L. The resolvent set $\rho(L)$ of L is the set of all complex numbers λ for which $\lambda I - L$ is invertible, i.e. $(\lambda I - L)^{-1}$ is a bounded linear operator on B. The family

$$\left\{R_\lambda \equiv (\lambda I - L)^{-1} : \lambda \in \rho(L)\right\}$$

of bounded linear operators is also called the resolvent of L. The complement of the resolvent set of L is called the spectrum of L denoted by $\sigma(L)$.

3.2 Hille-Yosida theorem

The necessary and sufficient condition for a given densely defined linear operator L (with domain $D(L)$) to be the infinitesimal generator of some strongly continuous contraction semigroup $(P_t)_{t \geq 0}$ is known as Hille-Yosida's theory of one-parameter semigroups.

h-yth1 **Theorem 3.2.1** (Hille-Yosida) A linear (unbounded) operator L is the infinitesimal generator of a strongly continuous semigroup of contractions on a Banach space B if and only if

1) L is closed and the $D(L)$ is dense in B, i.e. L is a densely defined closed operator.

2) $(0, +\infty) \subset \rho(L)$ and $\|R_\lambda\| \leq 1/\lambda$ for every $\lambda > 0$. In other words

$$\|(\lambda - L)x\| \geq \lambda \|x\|$$

for every $\lambda > 0$ and $x \in D(L)$.

We have proven the necessity of two conditions 1 and 2 (see Proposition propy03 3.1.4) and

$$(\lambda I - L)^{-1} = \int_0^\infty e^{-\lambda t} P_t dt \qquad \forall \lambda > 0,$$

namely, the resolvent of L is the Laplace transform of its semigroup $(P_t)_{t\geq 0}$.

We shall prepare a few lemmas before proceeding to prove Hille-Yosida theorem.

l:hy1 **Lemma 3.2.2** Let L satisfy the conditions 1 and 2 of Theorem h-yth1 3.2.1, and set $R_\lambda = (\lambda I - L)^{-1}$. Then

$$\lim_{\lambda\to\infty} \lambda R_\lambda x = x \qquad \forall x \in M\ .$$

Indeed, consider first those $x \in D(L)$. Then

$$\lambda R_\lambda x - x = L R_\lambda x = R_\lambda L x$$

so that

$$\begin{aligned} \|\lambda R_\lambda x - x\| &= \|R_\lambda L x\| \\ &\leq \frac{1}{\lambda}\|Lx\| \end{aligned}$$

as $\lambda \to \infty$. However $D(L)$ is dense in B and $\|\lambda R_\lambda x\| \leq 1$, therefore $\lambda R_\lambda x \to x$ as $\lambda \to \infty$ for every $x \in B$. That proves Claim 1.

For every $\lambda > 0$, the Yosida approximation of L is defined as

$$L_\lambda = \lambda L R_\lambda = \lambda^2 R_\lambda - \lambda I \ .$$

Note L_λ is a bounded linear operator for each $\lambda > 0$, and moreover, if $x \in D(L)$ then

$$L R_\lambda x = R_\lambda L x$$

and we have

ccc01 **Corollary 3.2.3** Let L satisfy the conditions 1 and 2 of Theorem h-yth1 3.2.1. Then

$$\lim_{\lambda\to\infty} L_\lambda x = Lx \qquad \forall x \in D(L) \ .$$

If T is a bounded linear operator on B, then its exponential e^T is given by the formula

$$\mathrm{e}^T(x) = \sum_{k=0}^{\infty} \frac{1}{k!}(T^k x)$$

which is again a bounded linear operator on B. Indeed

$$\begin{aligned} \|\mathrm{e}^T\| &\leq \sum_{k=0}^{\infty} \frac{1}{k!}\|T^k\| \\ &\leq \sum_{k=0}^{\infty} \frac{1}{k!}\|T\|^k = \mathrm{e}^{\|T\|} \ . \end{aligned}$$

If S and T are two bounded linear operator and if T and S commute, then

$$\mathrm{e}^{T+S} = \mathrm{e}^T \mathrm{e}^S \ .$$

In particular for a bounded linear operator T on a Banach space, then $(\mathrm{e}^{tT})_{t\in\mathbf{C}}$ is a commutative family of bounded linear operators, and

$$\|\mathrm{e}^{tT}(x) - x\| = \left\| \sum_{k=1}^{\infty} \frac{t^k}{k!}(T^k x) \right\|$$

$$\leq \sum_{k=1}^{\infty} \frac{t^k}{k!} \|T\|^k \|x\|$$
$$= \left(\mathrm{e}^{t\|T\|} - 1\right) \|x\|$$

so that

$$\|\mathrm{e}^{tT} - I\| \leq \mathrm{e}^{t\|T\|} - 1 \to 0 \qquad \text{as } t \to 0 \; .$$

Moreover

$$\left\| \frac{\mathrm{e}^{tT}(x) - x}{t} - Tx \right\| = \left\| \sum_{k=2}^{\infty} \frac{t^{k-1}}{k!} (T^k x) \right\|$$
$$= \left\| \sum_{k=0}^{\infty} \frac{t^{k+1}}{(k+2)!} (T^{k+2} x) \right\|$$
$$\leq t\|T\|^2 \left\| \sum_{k=0}^{\infty} \frac{t^k}{k!} (T^k x) \right\|$$
$$\leq t\|T\|^2 \mathrm{e}^{t\|T\|} \|x\|$$

Therefore $(\mathrm{e}^{tT})_{t\geq 0}$ is a strongly semigroup of bounded linear operators with infinitesimal generator T.

Since L_λ is bounded for any $\lambda > 0$, it is the infinitesimal generator of the strongly continuous semigroup

$$\mathrm{e}^{tL_\lambda} = \sum_{n=0}^{\infty} \frac{t^n}{n!} L_\lambda^n \; .$$

Moreover, since λI and R_λ commute, so that $\mathrm{e}^{tL_\lambda} = \mathrm{e}^{-\lambda t} \mathrm{e}^{t\lambda^2 R_\lambda}$ and therefore for any $t > 0$

$$\|\mathrm{e}^{tL_\lambda}\| \leq \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} \lambda^{2n} \|R_\lambda\|^n$$
$$= \mathrm{e}^{-\lambda t} \mathrm{e}^{\lambda^2 t \|R_\lambda\|}$$
$$\leq 1.$$

Hence for each $\lambda > 0$, $(\mathrm{e}^{tL_\lambda})_{t\geq 0}$ is a semigroup of contractions on B with infinitesimal generator L_λ. The next lemma shows that e^{tL_λ} converges as $\lambda \to \infty$. The limit shall be the strongly continuous semigroup of contractions with infinitesimal generator L.

l:hy2 **Lemma 3.2.4** Under the same assumption as in Lemma l:hy1 3.2.2. For any $\lambda, \mu > 0$

$$\left\|\mathrm{e}^{tL_\lambda}x - \mathrm{e}^{tL_\mu}x\right\| \leq t \left\|L_\lambda x - L_\mu x\right\| \ .$$

Therefore, for every $x \in D(L)$, $\mathrm{e}^{tL_\lambda}x$ converges as $\lambda \to \infty$ uniformly in t in any bounded interval.

Let us prove this claim. Since all bounded linear operators e^{tL_λ}, e^{tL_μ}, L_λ and L_μ commute with each other, consequently

$$\begin{aligned}
\mathrm{e}^{tL_\lambda}x - \mathrm{e}^{tL_\mu}x &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left(\mathrm{e}^{stL_\lambda}\mathrm{e}^{t(1-s)L_\mu}x\right)\mathrm{d}s \\
&= t\int_0^t \mathrm{e}^{stL_\lambda}\mathrm{e}^{t(1-s)L_\mu}\left(L_\lambda x - L_\mu x\right)\mathrm{d}s \ ,
\end{aligned}$$

together with the fact that $\|\mathrm{e}^{stL_\lambda}\| \leq 1$ and $\|\mathrm{e}^{t(1-s)L_\mu}\| \leq 1$, it follows thus that

$$\begin{aligned}
\left\|\mathrm{e}^{tL_\lambda}x - \mathrm{e}^{tL_\mu}x\right\| &\leq t\int_0^t \|\mathrm{e}^{stL_\lambda}\mathrm{e}^{t(1-s)L_\mu}\left(L_\lambda x - L_\mu x\right)\|\mathrm{d}s \\
&\leq t\|L_\lambda x - L_\mu x\| \ .
\end{aligned}$$

Let $x \in D(L)$. Then

$$\begin{aligned}
\left\|\mathrm{e}^{tL_\lambda}x - \mathrm{e}^{tL_\mu}x\right\| &\leq t\|L_\lambda x - L_\mu x\| \\
&= t\|L_\lambda x - Lx\| + t\|L_\mu x - Lx\| \ ,
\end{aligned}$$

by Corollary ccc01 3.2.3, it thus follows that $\mathrm{e}^{tL_\lambda}x$ converges as $\lambda \to \infty$ uniformly in t in any bounded interval. That proves the conclusion.

We are now in a position to complete the proof of the Hille-Yosida theorem 3.2.1. Let

$$P_t x \equiv \lim_{\lambda\to\infty} \mathrm{e}^{tL_\lambda} x \qquad \forall x \in D(L),\, t \geq 0\ . \tag{3.3}$$

Then $\|P_t x\| \leq 1$. Since $D(L)$ is dense in B, so that $\lim_{\lambda\to\infty} \mathrm{e}^{tL_\lambda} x$ exists for every $x \in B$. Moreover the convergence in (3.3) is uniform in t on any bounded interval, so that

$$\begin{aligned} P_{t+s}x &= \lim_{\lambda\to\infty} \mathrm{e}^{(t+s)L_\lambda} x = \lim_{\lambda\to\infty} \mathrm{e}^{tL_\lambda}\left(\mathrm{e}^{sL_\lambda} x\right) \\ &= \lim_{\lambda\to\infty} \mathrm{e}^{tL_\lambda}\left(\lim_{\lambda\to\infty} \mathrm{e}^{sL_\lambda} x\right) \\ &= P_t(P_s x) \end{aligned}$$

and

$$\begin{aligned} \lim_{t\downarrow 0} \|P_t x - x\| &= \lim_{t\downarrow 0} \| \lim_{\lambda\to\infty} \left(\mathrm{e}^{tL_\lambda}(x) - x\right) \| \\ &= \| \lim_{\lambda\to\infty} \lim_{t\downarrow 0} \left(\mathrm{e}^{tL_\lambda}(x) - x\right) \| \\ &= 0\ . \end{aligned}$$

Therefore $(P_t)_{t\geq 0}$ is a strongly continuous semigroup of contractions on B. Next we prove that L is the infinitesimal generator of $(P_t)_{t\geq 0}$. Let $(A, D(A))$ be the infinitesimal generator of $(P_t)_{t\geq 0}$. If $x \in D(L)$ then

$$\begin{aligned} P_t x - x &= \lim_{\lambda\to\infty} \left(\mathrm{e}^{tL_\lambda}(x) - x\right) = \lim_{\lambda\to\infty} \int_0^t \mathrm{e}^{sL_\lambda}(L_\lambda x)\mathrm{d}s \\ &= \int_0^t \lim_{\lambda\to\infty} \mathrm{e}^{sL_\lambda}(L_\lambda x)\mathrm{d}s = \int_0^t \lim_{\lambda\to\infty} \mathrm{e}^{sL_\lambda}(\lim_{\lambda\to\infty} L_\lambda x)\mathrm{d}s \\ &= \int_0^t P_s(Lx)\mathrm{d}s \end{aligned}$$

and therefore $x \in D(A)$ and $Ax = Lx$. Since 1 belongs to the resolvent sets both of A and L we therefore have

$$(I - A)D(L) = (I - L)D(L) = B$$

so that

$$D(A) = (I - A)^{-1}B = (I - L)^{-1}B$$
$$= D(L) .$$

Therefore $D(A) = D(L)$ and $L = A$.

The contraction semigroup $(P_t)_{t\geq 0}$ with infinitesimal generator L sometimes is denoted by e^{tL}, although e^{tL} is not necessarily given by power series.

co41 **Corollary 3.2.5** Let L be the infinitesimal generator of a strongly continuous, contraction semigroup $(P_t)_{t\geq 0}$ on a Banach space B, and let $L_\lambda = \lambda^2 R_\lambda - \lambda I$ (where $R_\lambda = (\lambda I - L)^{-1}$) be the Yosida approximation of L. Then

$$P_t x = \lim_{\lambda\to\infty} \mathrm{e}^{tL_\lambda} x$$

uniformly in t on any bounded interval.

co42 **Corollary 3.2.6** Under the same assumption in the previous corollary. The resolvent set $\rho(L) \supseteq \{\lambda : \mathrm{Re}\lambda > 0\}$ and

$$\|(\lambda I - L)^{-1}\| \leq \frac{1}{\mathrm{Re}\lambda}$$

for any λ such that such $\mathrm{Re}\lambda > 0$.

Proof. If $\mathrm{Re}\lambda > 0$ then

$$R_\lambda \equiv \int_0^\infty \mathrm{e}^{-\lambda t}(P_t x)\mathrm{d}t$$

is well defined bounded linear operator, which is $(\lambda I - L)^{-1}$. □

3.3 Contraction semigroups on Hilbert spaces

In this sub-section we specialize our study to a class of strongly continuous contraction semigroups of symmetric linear operators on a Hilbert space H.

A bounded linear operator T on a (real) Hilbert space H is called a symmetric operator if

$$\langle Tx, y\rangle = \langle x, Ty\rangle$$

where $\langle x, y\rangle$ is the inner product of H. The adjoint operator L^* of a densely defined linear operator L (with domain $D(L)$) on H is defined as the following: $x \in D(L^*)$ if

$$|\langle Ly, x\rangle| \leq C_x \|y\| \qquad \text{for every } y \in D(L)$$

for some non-negative constant C_x, and L^*x is the unique element in H (F. Riesz's representation) such that

$$\langle Ly, x\rangle = \langle y, L^*x\rangle \qquad \text{for every } y \in D(L)\ .$$

L^* is a closed linear operator on H. If $L^* = L$, then L is called a self-adjoint operator.

The fundamental tool in the study of self-adjoint operators is the spectral decomposition theorem. To appreciate this theorem, let us investigate an example which is in turn to be the general (function) model for self-adjoint operators.

Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space, and let $H = L^2(\Omega, \mathscr{F}, \mu)$. If ϕ is a real-valued measurable function on Ω, then we use T_ϕ to denote the multiplier operator

$$T_\phi x = \phi x \qquad \text{for any } x \in H$$

and

$$D(T_\phi) = \{x \in H : \phi x \in L^2(\Omega, \mathscr{F}, \mu)\}\ .$$

Then T_ϕ is a self-adjoint operator on H, with $\sigma(T_\phi)$ the essential range of ϕ. We note that if ϕ is an indicator function of a measurable subset A, then $T_{1_A} : L^2(\Omega, \mathscr{F}, \mu) \to L^2(A, \mathscr{F}, \mu)$ is a projection.

Given a real-valued measurable function ϕ we associate it with a right-continuous, increasing family $\{E_\lambda : \lambda \in \mathbb{R}\}$ of projections on H defined by $E_\lambda = T_{1_{\{\phi<\lambda\}}}$. Obviously $\lambda \to \langle E_\lambda x, x\rangle$ is increasing so that

$$\lambda \to \langle E_\lambda x, y\rangle$$

is right-continuous and has finite variation. Moreover $\lim_{\lambda\to-\infty} E_\lambda = 0$ and $\lim_{\lambda\to\infty} E_\lambda = I$. If $x \in H$ such that $\|x\| = 1$, then

$$\mathbf{P}_x(\mathrm{d}w) \equiv x(w)^2\mu(\mathrm{d}w)$$

is a probability measure, and

$$\begin{aligned}\langle E_\lambda x, x\rangle &= \int_X 1_{\{\phi<\lambda\}} x^2 \mathrm{d}\mu \\ &= \mathbf{P}_x\left\{\phi(w) < \lambda\right\}\end{aligned}$$

is the distribution function of ϕ under the probability measure $\mathbf{P}_x$. It is seen that ϕ is square-integrable with respect to $\mathbf{P}_x$ if and only if

$$\begin{aligned}\mathbf{E}_x|\phi|^2 &= \int_X |\phi|^2 \mathrm{d}\mathbf{P}_x = \int_X |\phi|^2 x^2 \mathrm{d}\mu \\ &= \int_{-\infty}^{\infty} |\lambda|^2 \mathrm{d}\langle E_\lambda x, x\rangle < +\infty \ ;\end{aligned}$$

which implies that $x \in D(T_\phi)$ if and only if

$$\int_{-\infty}^{\infty} \lambda^2 \mathrm{d}\langle E_\lambda x, x\rangle < +\infty \ .$$

Moreover

$$\begin{aligned}\int_{-\infty}^{\infty} \lambda \mathrm{d}\langle E_\lambda x, x\rangle = \mathbf{E}_x\left(\phi\right) &= \int_X \phi \mathrm{d}\mathbf{P}_x \\ &= \int_X \phi x^2 \mathrm{d}\mu\end{aligned}$$

and therefore

$$\langle T_\phi x, x\rangle = \int_{-\infty}^{\infty} \lambda \mathrm{d}\langle E_\lambda x, x\rangle \ .$$

By the polarization identity, we thus have

$$\langle T_\phi x, y\rangle = \int_{-\infty}^{\infty} \lambda \mathrm{d}\langle E_\lambda x, y\rangle \ .$$

The last equality shows that T_ϕ has a spectral decomposition

$$T_\phi = \int_{-\infty}^{\infty} \lambda \mathrm{d}E_\lambda = \int_{\sigma(T_\phi)} \lambda \mathrm{d}E_\lambda$$

as $\lambda \to E_\lambda$ increases only on the spectrum $\sigma(T_\phi)$, and

$$D(T_\phi) = \left\{ x : \int_{-\infty}^{\infty} \lambda^2 \mathrm{d}\langle E_\lambda x, x\rangle < +\infty \right\} \ .$$

One of the main achievement in Functional Analysis is the following spectral theorem, which claims that the above spectral decomposition holds for any self-adjoint operator.

Theorem 3.3.1 Let L be a self-adjoint operator on a Hilbert space H.

1) The spectrum $\sigma(L) \subset \mathbb{R}$.

2) There is a right-continuous and increasing family $\{E_\lambda : \lambda \in \mathbb{R}\}$ of projections in H such that

$$\lim_{\lambda \to -\infty} E_\lambda = 0 \quad \text{and} \quad \lim_{\lambda \to \infty} E_\lambda = I \ ;$$

$\lambda \to E_\lambda$ increases only on $\sigma(L)$,

$$D(L) = \left\{ x \in H : \int_{-\infty}^{\infty} \lambda^2 \mathrm{d}\langle E_\lambda x, x\rangle < +\infty \right\}$$

and

$$L = \int_{-\infty}^{+\infty} \lambda \mathrm{d}E_\lambda = \int_{\sigma(L)} \lambda \mathrm{d}E_\lambda$$

in the sense that

$$\langle Lx, y\rangle = \int_{-\infty}^{+\infty} \lambda \mathrm{d}\langle E_\lambda x, y\rangle \qquad \forall x \in D(L) \text{ and } y \in H,$$

where the right-hand side is the Riemann-Stieltjes integral.

A self-adjoint linear operator L is positive-definite if $\langle Lx, x\rangle \geq 0$ for every $x \in H$. Such a self-adjoint operator possesses a spectral decomposition

$$L = \int_0^{+\infty} \lambda \mathrm{d}E_\lambda \ .$$

Consider a self-adjoint linear operator L which is negative-definite, that is, $-L$ is positive-definite, and let $-L$ have the spectral decomposition

$$-L = \int_0^{+\infty} \lambda \mathrm{d}E_\lambda$$

or equivalently

$$L = \int_0^{+\infty} -\lambda \mathrm{d}E_\lambda \ .$$

Define

$$P_t = \int_0^{+\infty} \mathrm{e}^{-\lambda t} \mathrm{d}E_\lambda$$

which is a self-adjoint operator, and

$$\begin{aligned} |\langle P_t x, x\rangle| &\leq \int_0^{+\infty} \mathrm{e}^{-\lambda t} \mathrm{d}\langle E_\lambda x, x\rangle \\ &\leq \int_0^{+\infty} \mathrm{d}\langle E_\lambda x, x\rangle \\ &= \|x\|^2 \ . \end{aligned}$$

Therefore each P_t is a contraction on H, hence $(P_t)_{t\geq 0}$ is a strongly continuous contraction semigroup of symmetric operators on H with infinitesimal generator L. Conversely, it is obvious that the infinitesimal generator of a strongly continuous contraction semigroup of symmetric linear operators on a Hilbert space H is a negative-definite self-adjoint operator.

h-ys01th **Theorem 3.3.2** L is the infinitesimal generator of a strongly continuous contraction semigroup of symmetric linear operators on a Hilbert space H if and only if L is a negative-definite self-adjoint operator. If

$$L = \int_0^{+\infty} -\lambda \mathrm{d}E_\lambda \tag{3.4}$$

eqssu01

is the spectral decomposition of $-L$, then

$$\mathrm{e}^{tL} = \int_0^{+\infty} \mathrm{e}^{-\lambda t} \mathrm{d}E_\lambda$$

and for every $\alpha > 0$

$$R_\alpha = (\alpha - L)^{-1} = \int_0^{+\infty} \frac{1}{\alpha + \lambda} \mathrm{d}E_\lambda \ .$$

In general, if $-L$ is a positive-definite self-adjoint linear operator on H with spectral decomposition (3.4) then for any continuous function f on $[0, +\infty)$, $f(L)$ is a self-adjoint operator

$$f(L) = \int_0^{+\infty} f(-\lambda) \mathrm{d}E_\lambda$$

with domain

$$D(f(L)) = \left\{ x \in H : \int_0^{+\infty} f(-\lambda)^2 \mathrm{d}\langle E_\lambda x, x\rangle < +\infty \right\} \ .$$

The most important for our propose is the square root of $-L$ which can be defined as

$$\sqrt{-L} = \int_0^{+\infty} \sqrt{\lambda} \mathrm{d}E_\lambda$$

with domain

$$D(\sqrt{-L}) = \left\{ x \in H : \int_0^{+\infty} \lambda \mathrm{d}\langle E_\lambda x, x\rangle < +\infty \right\} \ .$$

$\sqrt{-L}$ is a positive-definite, self-adjoint operator. Obviously $D(L) \subset D(\sqrt{-L})$ and

$$\langle -Lx, y\rangle = \langle \sqrt{-L}x, \sqrt{-L}y\rangle \quad , \qquad \forall x \in D(L);\, y \in D(\sqrt{-L})$$

Let $(P_t)_{t\geq 0}$ be the semigroup generated by L:

$$P_t = \int_0^{+\infty} \mathrm{e}^{-\lambda t} \mathrm{d}E_\lambda$$

and set

$$\mathscr{E}^{(t)}(x,x) = \frac{1}{t}(x - P_t x, x) = \frac{1}{t}\left(\|x\|^2 - \|P_{t/2}x\|^2\right),\ t > 0, x \in H, \tag{3.5}$$

which is called an approximating form defined by (P_t). Then

$$\mathscr{E}^{(t)} = \frac{1}{t}\int_0^{+\infty} \left(1 - \mathrm{e}^{-\lambda t}\right) \mathrm{d}\langle E_\lambda x, x\rangle \ .$$

Since

$$\begin{aligned} \frac{d}{ds}\frac{1-\mathrm{e}^{-s}}{s} &= \frac{\mathrm{e}^{-s}s - 1 + \mathrm{e}^{-s}}{s^2} \\ &= -\frac{\mathrm{e}^{s} - 1 - s}{s^2\mathrm{e}^{s}} < 0 \quad \text{for } s > 0, \end{aligned}$$

$t \to \mathscr{E}^{(t)}(x,x)$ is decreasing and therefore

$$\lim_{t\downarrow 0} \mathscr{E}^{(t)}(x,x) = \sup_{t>0} \mathscr{E}^{(t)}(x,x)$$

exists, which will be denoted by $\mathscr{E}(x,x)$ $(\leq +\infty)$.

deq004 **Theorem 3.3.3** Let L be a negative-definite self-adjoint operator on Hilbert space H, and let $P_t = \mathrm{e}^{tL}$ be the semigroup generated by L. Then $x \in D(\sqrt{-L})$ if and only if $\mathscr{E}(x,x) < +\infty$. Moreover

$$\|\sqrt{-L}x\|^2 = \mathscr{E}(x,x) \tag{3.6}$$

for every $x \in D(\sqrt{-L})$. Similarly if we define an approximating form by (R_α) as

$$\mathscr{E}^{[\beta]}(x,y) = \beta(x - \beta R_\beta x, y),\ x, y \in H, \tag{3.7}$$

then $D(\sqrt{-L}) = \{x \in H : \sup_{\beta>0} \mathscr{E}^{[\beta]}(x,x) < \infty\}$ and

$$\mathscr{E}(x,x) = \|\sqrt{-L}x\|^2 = \lim_{\beta\to\infty} \mathscr{E}^{[\beta]}(x,x) \tag{3.8}$$

for every $x \in D(\sqrt{-L})$.

Proof. If $x \in D(\sqrt{-L})$ then

$$\int_0^{+\infty} \lambda \mathrm{d}\langle E_\lambda x, x\rangle < +\infty \ .$$

However

$$\frac{1-\mathrm{e}^{-s}}{s} \le 1 \quad \text{for all} \quad s \in (0,+\infty)$$

so that by Lebesgue's dominated convergence

$$\begin{aligned}
\lim_{t\downarrow 0} \frac{1}{2t}\left(\|x\|^2 - \|P_t x\|^2\right) &= \int_0^{+\infty} \lim_{t\downarrow 0} \frac{1-\mathrm{e}^{-2\lambda t}}{2t} \mathrm{d}\langle E_\lambda x, x\rangle \\
&= \int_0^{+\infty} \lambda \mathrm{d}\langle E_\lambda x, x\rangle \\
&= \|\sqrt{-L}x\|^2 \ .
\end{aligned}$$

On the other hand, if

$$\begin{aligned}
\lim_{t\downarrow 0} \frac{1}{2t}\left(\|x\|^2 - \|P_t x\|^2\right) &= \sup_{t>0} \frac{1}{2t} \int_0^{+\infty} \left(1-\mathrm{e}^{-2\lambda t}\right) \mathrm{d}\langle E_\lambda x, x\rangle \\
&< +\infty,
\end{aligned}$$

then by Fatou's lemma and the fact that $\frac{1-\mathrm{e}^{-2\lambda t}}{2t} > 0$, we have

$$\int_0^{+\infty} \lambda \mathrm{d}\langle E_\lambda x, x\rangle = \int_0^{+\infty} \lim_{t\downarrow 0} \frac{1-\mathrm{e}^{-2\lambda t}}{2t} \mathrm{d}\langle E_\lambda x, x\rangle$$

$$\leq \lim_{t\downarrow 0}\int_0^{+\infty}\frac{1-\mathrm{e}^{-2\lambda t}}{2t}\mathrm{d}\langle E_\lambda x, x\rangle < +\infty$$

that is $x \in D(\sqrt{-L})$. □

lowe004 **Corollary 3.3.4** Let L be a negative-definite self-adjoint operator L on Hilbert space H, and let $P_t = \mathrm{e}^{tL}$. Then $\mathscr{E}$ is lower semi-continuous on H, that is

$$\mathscr{E}(x,x) \leq \underline{\lim}_{n\to\infty}\mathscr{E}(x_n, x_n)$$

if $x_n \to x$ in H.

The lower semi-continuity follows from that $\mathscr{E}$ is the supremum of a family of continuous functions

$$\mathscr{E}(x,x) \equiv \sup_{t>0}\frac{1}{2t}\left(\|x\|^2 - \|P_t x\|^2\right) .$$

Definition 3.3.5 The quadratic form $(\mathscr{E}, D(\mathscr{E}))$ associated with a negative-definite self-adjoint operator L is defined by

$$\mathscr{E}(x,y) = \langle\sqrt{-L}x, \sqrt{-L}y\rangle , \quad x, y \in D(\mathscr{E})$$

where

$$D(\mathscr{E}) = D(\sqrt{-L}) .$$

The main advantage by considering the quadratic form $(\mathscr{E}, D(\mathscr{E}))$ instead of the (unbounded) self-adjoint operator L is that, as we have seen, $\mathscr{E}(x,x)$ is well-defined for every $x \in H$

$$\mathscr{E}(x,x) = \lim_{t\downarrow 0}\frac{1}{2t}\left(\|x\|^2 - \|P_t x\|^2\right)$$

and $\mathscr{E}(x,x)$ is finite if and only if $x \in D(\mathscr{E})$. The following proposition is evident since $\sqrt{-L}$ is a closed operator.

propgh01 **Proposition 3.3.6** Let L be a negative-definite self-adjoint operator on Hilbert space H, and let $(\mathscr{E}, D(\mathscr{E}))$ be the quadratic form associated with L. Define for $x \in D(\mathscr{E}), \alpha > 0$,

$$\begin{aligned}\mathscr{E}_\alpha(x,x) &= \alpha\|x\|^2 + \mathscr{E}(x,x)\\ &= \alpha\|x\|^2 + \|\sqrt{-L}x\|^2.\end{aligned}$$

Then $D(\mathscr{E})$ is a Hilbert space with respect to $\mathscr{E}_\alpha$.

We now define the concept of symmetric forms on a Hilbert space and prove that a symmetric form corresponds to a strongly continuous contraction resolvent, or equivalently a strongly continuous contraction semigroup and a non-positive definite self-adjoint operator.

Definition 3.3.7 A densely defined bilinear form $(\mathscr{E}, D(\mathscr{E}))$ on H is called a symmetric form on H if it is symmetric, non-negative definite and $D(\mathscr{E})$ is a Hilbert space with respect to the inner product $\mathscr{E}_\alpha$.

Theorem 3.3.8 If $(\mathscr{E}, D(\mathscr{E}))$ is a symmetric form on H, then there exists a unique strongly continuous symmetric contraction resolvent $(R_\alpha, \alpha > 0)$ such that

$$\mathscr{E}_\alpha(R_\alpha x, y) = \langle x, y\rangle,\ \ \alpha > 0, x \in H, y \in D(\mathscr{E}). \tag{3.9}$$

Proof. Fix $x \in H$ and $\alpha > 0$. Then $y \mapsto \langle x, y\rangle$ is a bounded linear operator on Hilbert space $(D(\mathscr{E}), \mathscr{E}_\alpha)$ since

$$|\langle x, y\rangle| \le \|x\|\|y\| \le \frac{1}{\sqrt{\alpha}}\|x\| \cdot \|y\|_{\mathscr{E}_\alpha},$$

where $\|\cdot\|_{\mathscr{E}_\alpha} = \sqrt{\mathscr{E}_\alpha(\cdot,\cdot)}$. By Riesz representation theorem, there exists a unique element in $D(\mathscr{E})$, denoted by $R_\alpha x$, such that

$$\mathscr{E}_\alpha(R_\alpha x, y) = \langle x, y\rangle,\ \ y \in D(\mathscr{E}). \tag{3.10}$$

Clearly it follows that

$$\alpha\|R_\alpha x\|^2 \leq \mathscr{E}_\alpha(R_\alpha x, R_\alpha x) = \langle x, R_\alpha x\rangle \leq \|x\|\|R_\alpha x\|$$

and hence we have $\|\alpha R_\alpha x\| \leq \|x\|$, i.e., αR_α is a contraction. Moreover $(R_\alpha, \alpha > 0)$ is a strongly continuous symmetric contraction resovent on H. In fact, taking $\alpha > \beta > 0$ and $x, y \in H$,

$$\begin{aligned}\mathscr{E}_\alpha(R_\alpha x, y) &= (x, y) = \mathscr{E}_\beta(R_\beta x, y)\\ &= \mathscr{E}_\alpha(R_\beta x, y) + (\beta - \alpha)\langle R_\beta x, y\rangle\\ &= \mathscr{E}_\alpha(R_\beta x + (\beta - \alpha)R_\alpha R_\beta x, y)\end{aligned}$$

and it implies the resolvent equation

$$R_\alpha - R_\beta + (\alpha - \beta)R_\alpha R_\beta = 0. \tag{3.11}$$

To verify the strong continuity, we prove the following inequality first. For $x \in D(\mathscr{E})$,

$$\begin{aligned}\alpha\|\alpha R_\alpha x - x\|^2 &\leq \mathscr{E}_\alpha(\alpha R_\alpha x - x, \alpha R_\alpha x - x)\\ &= \alpha(x, \alpha R_\alpha x - x) + \mathscr{E}(x, x) \leq \mathscr{E}(x, x),\end{aligned}$$

since, by contraction property, $\langle x, \alpha R_\alpha x\rangle \leq \langle x, x\rangle$. This implies $\alpha R_\alpha x \to x$ as $\alpha \to \infty$ for $x \in D(\mathscr{E})$ and hence for $x \in H$ due to denseness. □

3.4 Markovian property and Dirichlet forms

In the previous section, we have proven that in any abstract Hilbert space, there is essentially a one-to-one correspondence among strongly continuous contraction semigroups, strongly continuous contraction resolvents, non-positive definite self-adjoint operators and symmetric forms. However what we are really interested is a symmetric form on L^2 space with Markovian property, which we call a Dirichlet form.

Let $(E, \mathscr{E})$ be a measurable space with a σ-finite measure m, and for $p \geq 1$

$$L^p(E;m) = \{f \in \mathscr{E} : \int_E |f|^p dm < \infty\}$$

with so-called L^p-norm

$$\|f\|_{L^p} = \left(\int_E |f|^p dm\right)^{\frac{1}{p}}.$$

Of course two functions are equal if they are equal m-a.e. Let $(\mathscr{E}, D(\mathscr{E}))$ be a symmetric form on $L^2(E;m)$ with strongly continuous symmetric contraction semigroup (P_t), strongly continuous symmetric contraction resolvent (R_α) and infinitesimal generator $(L, D(L))$. Usually we use $\mathscr{F}$ instead of $D(\mathscr{E})$ in this case. For two measurable functions u, v on E, v is called a **normal contraction** of u if, for some version of u, v,

$$|u(x)| \leq |v(x)|, \ \forall x \in E;$$
$$|u(x) - u(y)| \leq |v(x) - v(y)|, \ \forall x, y \in E.$$

Definition 3.4.1 A symmetric form $(\mathscr{E}, \mathscr{F})$ is called **Markovian** if any normal contraction operates on $\mathscr{E}$, more precisely, if $u \in \mathscr{F}$ and v is a normal contraction of u, then $v \in \mathscr{F}$ and $\mathscr{E}(v,v) \leq \mathscr{E}(u,u)$. A Markovian symmetric form on $L^2(E;m)$ is called a **Dirichlet form on** $L^2(E;m)$.

For convenience, write $\|u\|_{\mathscr{E}} = \sqrt{\mathscr{E}(u,u)}$ for $u \in \mathscr{F}$, which is a semi-norm.

Theorem 3.4.2 A Dirichlet form $\mathscr{E}$ on $L^2(E;m)$ possesses the following properties:

(a) if $u \in \mathscr{F}$, then $|u| \in \mathscr{F}$ and $\mathscr{E}(|u|, |u|) \leq \mathscr{E}(u,u)$.

(b) if $u, v \in \mathscr{F}$, then $u \wedge v$, $u \vee v$, $u \wedge 1 \in \mathscr{F}$.

(c) if $u, v \in \mathscr{F} \cap L^\infty(E; m)$, then $uv \in \mathscr{F}$ and

$$\sqrt{\mathscr{E}(uv, uv)} \leq \|u\|_\infty \sqrt{\mathscr{E}(v, v)} + \|v\| \sqrt{\mathscr{E}(u, u)}.$$

(d) if $u \in \mathscr{F}$ and $u_n = ((-n) \vee u) \wedge n$, then $u_n \in \mathscr{F}$ and $u_n \to u$ in $\mathscr{E}_1$-norm.

(e) if $u \in \mathscr{F}$ and $u^{(\varepsilon)} = u - ((-\varepsilon) \vee u) \wedge \varepsilon$ for $\varepsilon > 0$, then $u^{(\varepsilon)} \in \mathscr{F}$ and $u^{(\varepsilon)} \to u$ in $\mathscr{E}_1$-norm as $\varepsilon \downarrow 0$.

Proof. (a), then (b), is obvious since $| \cdot |$ is a normal contraction. For (c), since $\|u\|_\infty v + \|v\|_\infty u$ is a normal contraction of uv, it follows that $uv \in \mathscr{F}$ and $\|uv\|_{\mathscr{E}} \leq \| \|u\|_\infty v + \|v\|_\infty u\|_{\mathscr{E}}$. Then by the triangle inequality, (c) holds.

(d) Clearly u_n is a normal contraction of u and hence $\mathscr{E}_1(u_n, u_n) \leq \mathscr{E}_1(u, u)$.Then for any $v \in L^2(E; m)$,

$$\mathscr{E}_1(u_n, R_1 v) = (u_n, v) \to (u, v) = \mathscr{E}_1(u, R_1 v).$$

Since

$$\mathscr{E}(\beta R_\beta x - x, \beta R_\beta x - x) = \mathscr{E}(x, x) - \beta(x, x - \beta R_\beta x),$$

it follows from Theorem deg004 3.3.3 that $R_1(L^2(E; m))$ is dense in $\mathscr{F}$ with respect to the norm $\mathscr{E}_1$. Therefore $\mathscr{E}_1(u_n, v) \to \mathscr{E}_1(u, v)$ for any $v \in \mathscr{F}$ and $\mathscr{E}_1(u_n - u, u_n - u) \leq 2\mathscr{E}_1(u, u) - 2\mathscr{E}_1(u_n, u) \to 0$. The proof of (e) is similar. □

Exercise 3.1 Assume that $u_n, u \in \mathscr{F}$, $u_n \to u$ with respect to $\mathscr{E}_1$-norm and ϕ is a real function such that $\phi(0) = 0$, $|\phi(t) - \phi(s)| \leq |t - s|$ for any $t, s \in \mathbf{R}$. Prove that $\phi(u_n), \phi(u) \in \mathscr{F}$ and $\phi(u_n) \to \phi(u)$ weakly with respect to $\mathscr{E}_1$-norm. If, in addition, $\phi(u) = u$, then the convergence is in norm.

Abounded linear operator S on $L^2(E;m)$ is called **Markovian** if $0 \le Su \le 1$ a.e. whenever $u \in L^2(E;m)$, $0 \le u \le 1$ a.e. This implies S is positive, i.e., $Su \ge 0$ a.e. for $u \ge 0$ a.e. The semigroup (P_t) is **Markovian** if P_t is Markovian for every $t > 0$ and the resolvent (R_α) is **Markovian** if αR_α is Markovian for every $\alpha > 0$. It is evident that the semigroup (P_t) is Markovian if and only if its resolvent (R_α) is Markovian.

t:0430-5 **Theorem 3.4.3** $\mathscr{E}$ is Markovian if and only if its resolvent (R_α) is Markovian.

Proof. Let's prove necessity first. Fix $\alpha > 0$ and $u \in L^2(E;m)$ such that $0 \le u \le 1$ a.e. Define a quadratic form ψ on $\mathscr{F}$ by

$$\psi(v) = \mathscr{E}(v,v) + \alpha\langle v - u/\alpha, v - u/\alpha\rangle,\ v \in \mathscr{F}. \tag{3.12}$$

Since $\mathscr{E}_\alpha(R_\alpha u, v) = \langle u, v\rangle$,

$$\psi(v) = \psi(R_\alpha u) + \mathscr{E}_\alpha(R_\alpha u - v, R_\alpha u - v). \tag{3.13}$$

Then $R_\alpha u$ is the unique element in $\mathscr{F}$ minimizing ψ. Let

$$w = \frac{1}{\alpha}\cdot(0 \vee \alpha R_\alpha u) \wedge 1 = (0 \vee R_\alpha u) \wedge \frac{1}{\alpha}.$$

Then w is a normal contraction of $R_\alpha u$ and it follows that $w \in \mathscr{F}$ and $\mathscr{E}(w,w) \le \mathscr{E}(R_\alpha u, R_\alpha u)$. On the other hand, since $0 \le u \le 1$ a.e., we have $|w - u/\alpha| \le |R_\alpha u - u/\alpha|$ and easily

$$\|w - u/\alpha\|_{L^2} \le \|R_\alpha u - u/\alpha\|_{L^2}.$$

Combining these, we have $\psi(w) \le \psi(R_\alpha u)$. By the uniqueness, $w = R_\alpha u$ or $0 \vee \alpha R_\alpha u \wedge 1 = \alpha R_\alpha u$, which implies $0 \le \alpha R_\alpha u \le 1$ a.e. This proves that (R_α) is Markovian.

The sufficiency is more difficult. The proof below is cited from [?]. Assume that (αR_α) is Markovian. For any $u \in \mathscr{F}$, let v is a normal contraction of u. We need to show that $v \in \mathscr{F}$ and $\mathscr{E}(v,v) \le \mathscr{E}(u,u)$. Indeed, by the approximating form, it suffices to prove that for any $\beta > 0$,

$$\langle v, v - \beta R_\beta v\rangle \le \langle u, u - \beta R_\beta u\rangle. \tag{3.14}$$

e:0430-1

Since the simple functions are dense in $L^2(E;m)$ we may assume that

$$u = \sum_{i=1}^n a_i 1_{A_i},$$

with $\{A_i\}$ disjoint and $m(A_i) < \infty$. Since v is a normal contraction, v is constant on every A_i, i.e., there exist $\{b_i\}$ such that $v = \sum_{i=1}^n b_i 1_{A_i}$ and (3.14) is equivalent to

$$\sum_{i,j} b_i b_j \langle 1_{A_i} - \beta R_\beta 1_{A_i}, 1_{A_j}\rangle \le \sum_{i,j} a_i a_j \langle 1_{A_i} - \beta R_\beta 1_{A_i}, 1_{A_j}\rangle. \tag{3.15}$$

e:0430-2

Using identity $xy = \frac{1}{2}((x+y)^2 - x^2 - y^2)$ for real $x, y \in \mathbf{R}$, we get

$$\begin{aligned}
&\sum_{i,j} b_i b_j \langle 1_{A_i} - \beta R_\beta 1_{A_i}, 1_{A_j}\rangle \\
&\quad = \sum_{i,j} b_i b_j \langle 1_{A_i}, 1_{A_j}\rangle - \sum_{i,j} b_i b_j \langle \beta R_\beta 1_{A_i}, 1_{A_j}, 1_{A_j}\rangle \\
&\quad = \sum_i b_i^2 \langle 1 - \beta R_\beta 1, 1_{A_i}\rangle + \frac{1}{2}\sum_{i,j} (b_i - b_j)^2 \langle \beta R_\beta 1_{A_i}, 1_{A_j}\rangle.
\end{aligned}$$

The normal contraction property implies that

$$|b_i| \le |a_i|, \ |b_i - b_j| \le |a_i - a_j|, \ i, j = 1, \cdots, n.$$

Then (3.15) follows immediately from the Markovian property: $\beta R_\beta 1 \le 1$ and $\beta R_\beta 1_{A_i} \ge 0$. $\square$

In particular, $u^+ \wedge 1$ is a normal contraction of u and is called a unit contraction. The first half in the proof of Theorem t:0430-5 3.4.3 proves that if every unit contraction operates on $\mathscr{E}$, then (αR_α) is Markovian and hence every normal contraction operates on $\mathscr{E}$.

3.5 Capacity

In this section when we talk about an equality or inequality concerning functions in $L^2(E;m)$ without referring to any measure we mans m-almost everywhere.

For an open subset A of E, if $\mathscr{L}_A = \{u \in \mathscr{F} : u \geq 1 \text{ on } A\}$ is non-empty, we say A has finite capacity and define the capacity of A as

$$\mathrm{C}(A) = \inf\{\mathscr{E}_1(u,u) : u \in \mathscr{L}_A\}. \tag{3.16}$$

Since $\mathscr{L}_A$ is a closed convex subset in a Hilber space $(\mathscr{F}, \mathscr{E}_1)$, there exists a unique element in $\mathscr{L}_A$, denoted by e_A and called the equilibrium potential of A, such that $\mathrm{C}(A) = \mathscr{E}_1(\mathrm{e}_A, \mathrm{e}_A)$. Since $\mathrm{e}_A^+ \wedge 1 \in \mathscr{L}_A$ is a normal contraction of e_A, $\mathscr{E}_1(\mathrm{e}_A^+ \wedge 1, \mathrm{e}_A^+ \wedge 1) \leq \mathscr{E}_1(\mathrm{e}_A, \mathrm{e}_A)$ and hence $\mathrm{e}_A^+ \wedge 1 = \mathrm{e}_A$ by uniqueness. It follows that $0 \leq \mathrm{e}_A \leq 1$ and $\mathrm{e}_A = 1$ on A.

l:0503-1 **Lemma 3.5.1** $\mathrm{e}_A = 1$ on A and for any $u \in \mathscr{F}$ with $u \geq 0$ on A, $\mathscr{E}_1(u, \mathrm{e}_A) \geq 0$. Actually e_A is the unique element in $\mathscr{L}_A$ satisfying the property above.

Proof. For such u, $\lambda u + \mathrm{e}_A \in \mathscr{L}_A$ for any $\lambda \geq 0$. Then

$$\begin{aligned}\mathscr{E}_1(\mathrm{e}_A, \mathrm{e}_A) &\leq \mathscr{E}_1(\lambda u + \mathrm{e}_A, \lambda u + \mathrm{e}_A) \\ &= \lambda^2 \mathscr{E}_1(u,u) + 2\lambda \mathscr{E}_1(u, \mathrm{e}_A) + \mathscr{E}_1(\mathrm{e}_A, \mathrm{e}_A)\end{aligned}$$

and it follows that $\mathscr{E}_1(u, \mathrm{e}_A) \geq 0$. Conversely if $v \in \mathscr{L}_A$ satisfies $v = 1$ on A and $\mathscr{E}_1(v, u) \geq 0$ for each $u \in \mathscr{F}$ with $u \geq 0$ on A, then for any $w \in \mathscr{L}_A$,

$w - v \geq 0$ on A, and hence

$$\mathscr{E}_1(w, w) = \mathscr{E}_1(w - v + v, w - v + v) \geq \mathscr{E}_1(v, v)$$

namely v minimizes $\mathscr{E}_1$ in $\mathscr{L}_A$ and it implies that $v = \mathrm{e}_A$. □

It is seen from this lemma that $\mathscr{E}_1(u, \mathrm{e}_A) = 0$ if $u = 0$ on A or $\mathscr{E}_1(u, \mathrm{e}_A) = \mathrm{C}(A)$ if $u = 1$ on A.

l:0503-2 **Lemma 3.5.2** Assume that A, B are open and $A \subset B$. Then $\mathrm{C}(A) \leq \mathrm{C}(B)$ and $\mathrm{e}_A \leq \mathrm{e}_B$.

Proof. The first claim is obvious. For the second claim, it suffices to prove that $(\mathrm{e}_B - \mathrm{e}_A)^+ + \mathrm{e}_A = \mathrm{e}_B$. Denote the left side by u. Then $u \in \mathscr{L}_B$, and, using Markovian property and Lemma 3.5.1, we have

$$\begin{aligned}\mathscr{E}_1(u, u) &= \mathscr{E}_1((\mathrm{e}_B - \mathrm{e}_A)^+, (\mathrm{e}_B - \mathrm{e}_A)^+) + 2\mathscr{E}_1(\mathrm{e}_A, (\mathrm{e}_B - \mathrm{e}_A)^+) + \mathscr{E}_1(\mathrm{e}_A, \mathrm{e}_A) \\ &\leq \mathscr{E}_1(\mathrm{e}_B - \mathrm{e}_A, \mathrm{e}_B - \mathrm{e}_A) + \mathscr{E}_1(\mathrm{e}_A, \mathrm{e}_A) \\ &= \mathscr{E}_1(\mathrm{e}_B, \mathrm{e}_B) - 2\mathscr{E}_1(\mathrm{e}_B - \mathrm{e}_A, \mathrm{e}_A) = \mathscr{E}_1(\mathrm{e}_B, \mathrm{e}_B).\end{aligned}$$

It implies that $u = \mathrm{e}_B$. □

p:0504-1 **Proposition 3.5.3** The capacity of open sets has the following properties which make it a Choquet capacity on E.
(1) For open $A \subset B$, $\mathrm{C}(A) \leq \mathrm{C}(B)$;
(2) For open $A_n \uparrow A$, $\mathrm{C}(A_n) \uparrow \mathrm{C}(A)$;
(3) For open A, B, $\mathrm{C}(A \cup B) + \mathrm{C}(A \cap B) \leq \mathrm{C}(A) + \mathrm{C}(B)$.

Proof. (1) has been shown. (2) It suffices to show that $\lim_n \mathscr{E}_1(\mathrm{e}_{A_n}, \mathrm{e}_{A_n}) \geq \mathrm{C}(A)$, where the sequence on the left is increasing. Since $\mathscr{E}_1(\mathrm{e}_{A_n}, \mathrm{e}_{A_n}) \leq \mathscr{E}_1(\mathrm{e}_A, \mathrm{e}_A)$, by Banach-alaoglu Theorem, there exists a subsequence of e_{A_n}, say itself, such that its Cesaro mean $\frac{1}{n}\sum_{k=1} \mathrm{e}_{A_k}$ converges to some $u \in \mathscr{F}$ in

$\mathscr{E}_1$-norm and also almost surely. Due to the fact that e_{A_n} is increasing, $u \geq 1$ on A and hence

$$\begin{aligned}\|\mathrm{e}_A\|_{\mathscr{E}_1} \leq \|u\|_{\mathscr{E}_1} &= \lim \left\| \frac{1}{n} \sum_{k=1}^{n} \mathrm{e}_{A_k} \right\|_{\mathscr{E}_1} \\ &\leq \lim_n \frac{1}{n} \sum_{k=1}^{n} \|\mathrm{e}_{A_k}\|_{\mathscr{E}_1} \\ &= \lim_n \|\mathrm{e}_{A_n}\|_{\mathscr{E}_1},\end{aligned}$$

where $\|u\|_{\mathscr{E}_1} = \sqrt{\mathscr{E}_1(u,u)}$, the $\mathscr{E}_1$-norm on $\mathscr{F}$.

(3) We prove

$$\mathrm{e}_A + \mathrm{e}_B - \mathrm{e}_{A\cup B} - \mathrm{e}_{A\cap B} \geq 0 \tag{3.17}$$

e:0503-1

on $A \cup B$. Indeed, since $\mathrm{e}_B \geq \mathrm{e}_{A\cap B}$ by Lemma 3.5.2 and $\mathrm{e}_A = \mathrm{e}_{A\cup B}$ on A, (3.17) is true on A. Similarly (3.17) is true on B.

Now by Lemma 3.5.1, $\mathscr{E}_1(\mathrm{e}_A + \mathrm{e}_B - \mathrm{e}_{A\cup B} - \mathrm{e}_{A\cap B}, \mathrm{e}_{A\cup B}) \geq 0$ and the conclusion follows from the remark below Lemma 3.5.1. □

Now for any subset $A \subset E$, define the capacity of A by

$$\mathrm{C}(A) = \inf\{\mathrm{C}(B) : B \text{ open and } A \subset B\}. \tag{3.18}$$

When A is open, this definition is consistent with that above. Clearly a set A is capacity zero if and only if there exist open subsets $B_n \downarrow$ such that $A \subset \bigcap_n B_n$ and $\mathrm{C}(B_n) \downarrow 0$.

Theorem 3.5.4 The capacity on all subsets defined above satisfies (1)-(3) in Proposition 3.5.3.

Proof. (1) is obvious. (2) Let $A_n \uparrow A$. It suffices to show that $\lim_n \mathrm{C}(A_n) \geq \mathrm{C}(A)$. Take open B_n such that $B_n \supset A_n$ and $\mathrm{C}(B_n) - \mathrm{C}(A_n) < \varepsilon/2^n$. Set

$G_n = B_1 \cup \cdots \cup B_n$. We may verify by induction that

$$C(G_n) - C(A_n) < \sum_{k=1}^{n} \varepsilon/2^k < \varepsilon.$$

Since $\bigcup G_n \supset A$,

$$\lim_n C(A_n) + \varepsilon \geq \lim_n C(G_n) = C(\bigcup_n G_n) \geq C(A).$$

By the arbitrariness of ε, we have the conclusion. The proof of (3) is easy and left as an exercise. □

Exercise 3.2 If B is Borel subset of E and $C(B) = 0$, then $m(B) = 0$.

3.6 Regularity and quasi-continuous version

Definition 3.6.1 A function u on E is called **quasi-continuous** if for any $\varepsilon > 0$, there exists a closed set F such that $C(F^c) < \varepsilon$ and u is continuous on F. A sequence $\{F_n\}$ of closed sets increasing to E is called a **nest** on E if $C(E \setminus F_n) \downarrow 0$. A nest $\{F_n\}$ is m-**regular** if $\mathrm{supp}(1_{F_n} \cdot m) = F_n$ for each n.

Clearly u is quasi-continuous if and only if there exists a nest $\{F_n\}$ such that u is continuous on each F_n. We now assume that the Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2(E; m)$ is **regular** in the sense of Fukushima, i.e., E is locally compact with countable base, $C_\infty \cap \mathscr{F}$ is dense both in C_∞ under the uniform norm and in $\mathscr{F}$ under $\mathscr{E}_1$-norm.

Exercise 3.3 If $(\mathscr{E}, \mathscr{F})$ is regular, then m must be a Radon measure.

Proof. Let K be a compact subset of E. Take a relatively compact open set $G \supset K$. There exists a continuous function $u \geq 0$ such that $u = 1$ on K and $u = 0$ on G^c. Then $u \in C_\infty(E)$ and by regularity there exists

$v \in C_\infty \cap \mathscr{F}$, such that $\|u - v\|_\infty < \frac{1}{2}$. It is clear that $v \geq \frac{1}{2}$ on K. Using Markovian property of $\mathscr{E}$, $v^+ \wedge 1 \in \mathscr{F}$ and it is no less than $\frac{1}{2}$ on K also. Hence $\infty > \langle v^+ \wedge 1, v^+ \wedge 1\rangle \geq \frac{1}{4} m(K)$, i.e., $m(K) < \infty$. □

Exercise 3.4 For a given nest $\{F_n\}$ on E, let $F_n' = \mathrm{supp}(1_{F_n} \cdot m)$. Then $\{F_n'\}$ is an m-regular nest.

Exercise 3.5 If u is quasi-continuous, then $u \geq 0$ a.e. implies that $u \geq 0$ except on a capacity zero set.

l:capineq **Lemma 3.6.2** For $u \in C_\infty \cap \mathscr{F}$ and $\lambda > 0$,

$$\mathrm{C}(\{|u| \geq \lambda\}) \leq \frac{1}{\lambda^2} \mathscr{E}_1(u, u). \tag{3.19}$$ e:0504-1

Proof. Since $\lambda^{-1} u \geq 1$ on $\{|u| \geq \lambda\}$ which is open, (3.19) is immediate. □

Theorem 3.6.3 If $(\mathscr{E}, \mathscr{F})$ is a regular Dirichlet form on $L^2(E; m)$, then each element $u \in \mathscr{F}$ has a quasi-continuous version.

Proof. Take $u_n \in C_\infty(E) \cap \mathscr{F}$ such that $\|u - u_n\|_{\mathscr{E}_1} \to 0$. Then there exists a subsequence of $\{u_n\}$, say itself, such that

$$\|u_{n+1} - u_n\|_{\mathscr{E}_1} \leq \frac{1}{2^n}.$$

By Lemma 3.6.2,

$$\mathrm{C}\left(\left\{|u_{n+1} - u_n| \geq \frac{1}{n^2}\right\}\right) \leq \frac{n^4}{2^{2n}}.$$

Then by the countable subadditivity, we have

$$\mathrm{C}\left(\bigcup_{n\geq k} \{|u_{n+1} - u_n| \geq n^{-2}\}\right) \leq \sum_{n \geq k} \frac{n^4}{2^{2n}}.$$

Set

$$F_k = \left(\bigcup_{n\geq k} \{|u_{n+1} - u_n| \geq n^{-2}\}\right)^c = \bigcap_{n\geq k} \{|u_{n+1} - u_n| < n^{-2}\}$$

for $k \geq 1$. Then $\lim_k \mathrm{C}(F_k^c) = 0$ and $|u_{n+1}(x) - u_n(x)| < n^{-2}$ holds for all $x \in F_k$ and $n \geq k$. Hence u_n converges to some function v uniformly on F. It follows that $u_n \to v$ pointwisely on $\bigcup_n F_n$. On the other hands, $u_n \to u$ a.e. m on E. This means $u = v$ a.e. m on $\bigcup_n F_n$. However the complement of $\bigcup_n F_n$ has capacity zero and hence has measure zero. It implies that $u = v$ a.e. on E and v is clearly quasi-continuous. □

Chapter 4

Symmetric Markov processes

4.1 Symmetric Borel right processes

Let X be a Borel right process on E with transition semigroup (P_t).

Definition 4.1.1 Let m be a σ-finite measure on $(E, \mathscr{E})$. The process X is called symmetric with respect to m if for any non-negative measurable functions f, g and $t \geq 0$,

$$\int g(x) P_t f(x) m(dx) = \int f(x) P_t g(x) m(dx).$$

If we write the inner product of $f, g \in L^2(E; m)$ as $\langle f, g \rangle$, this means

$$\langle P_t f, g \rangle = \langle f, P_t g \rangle.$$

It follows that $m \in \text{Exc}$.

Lemma 4.1.2 If $\xi \in \text{Exc}$ and $p \geq 1$, then (P_t) may be extended to a contraction semigroup on $L^p(E; \xi)$.

Proof. For $f, g \in \mathscr{E}$ with $f = g$ ξ-a.e., since $\xi|P_t f - P_t g| \leq \xi P_t|f - g| \leq \xi|f - g| = 0$, $P_t f = P_g$ ξ-a.e. Hence for any $f \in L^p(E; \xi)$ with $p \geq 1$, $P_t f$

does not depend on any particular version of f. By Hölder's inequality

$$
\begin{aligned}
|P_t f(x)| &= \left| \int_E P_t(x,dy) f(y) \right| \\
&\leq \int_E P_t(x,dy)|f(y)| \leq \left(\int_E P_t(x,dy)|f(y)|^p \right)^{1/p}.
\end{aligned}
$$

Hence we have

$$
\|P_t f\|_{L^p}^p = \int_E |P_t f(x)|^p \xi(dx) \leq \int_E P_t(|f|^p)(x)\xi(dx) = \xi P_t(|f|^p) \leq \|f\|_{L^p}^p,
$$

i.e., (P_t) is also a contraction semigroup on $L^p(E;\xi)$. □

Theorem 4.1.3 (P_t) is a strongly continuous contraction semigroup on $L^2(E;m)$.

Proof. Take $\alpha > 0$ and set

$$
D = \{U^\alpha f : \ f \in b\mathscr{E} \cap L^1(E;m)\} \subset L^2(E;m).
$$

Then D is dense in $L^2(m)$. Indeed, it suffices to show that if $g \in L^2(E;m)$ and $\langle g, U^\alpha f\rangle = 0$ for all $f \in b\mathscr{E} \cap L^1(E;m)$, then $g = 0$ a.e. By the resolvent equation, it follows that $\langle g, U^\beta f\rangle = 0$ for all $\beta > 0$. Choose $h = U^1 k$ where $k \in b\mathscr{E} \cap L^1(E;m)$ and strictly positive. Then for any bounded $f \in C(E)$, $t \mapsto f(X_t)h(X_t)$ is right continuous and hence $\beta U^{\beta+1}(fh) \longrightarrow fh$ a.e. as $\beta \to \infty$. However $\langle g, \beta U^{\beta+1}(fh)\rangle = 0$. Since h is 1-excessive,

$$
\beta|g \cdot U^{\beta+1}(fh)| \leq \beta|g|U^{\beta+1}|fh| \leq \|f\|_\infty |g| \beta U^{\beta+1} h \leq \|f\|_\infty |g| h.
$$

It follows from the dominated convergence theorem that

$$
\langle g, f\rangle_{h\cdot m} = \langle g, fh\rangle = \lim_{\beta\to\infty} \langle g, \beta U^{\beta+1}(fh)\rangle = 0.
$$

Since $h \cdot m$ is a finite measure, $C(E)$ is dense in $L^2(E; h\cdot m)$ and then $g = 0$ a.e. m.

We now prove the strong continuity. Fix $\alpha > 0$.Let $u = U^\alpha f \in D$. Then

$$u(x) = \int_0^t e^{-\alpha s} P_s f(x) ds + e^{-\alpha t} P_t u(x)$$

and, as $t \downarrow 0$,

$$\begin{aligned}\|P_t u - u\|_{L^2} &\le \left\|e^{-\alpha t} P_t u - u\right\|_{L^2} \\ &\le (1 - e^{-\alpha t})\|u\|_{L^2} + t\|f\|_{L^2} \to 0.\end{aligned}$$

For any $u \in L^2(E; m)$, take $u_n \in D$ such that $u_n \to u$ in L^2. Then

$$\begin{aligned}\|P_t u - u\|_{L^2} &\le \|P_t u - P_t u_n\|_{L^2} + \|P_t u_n - u_n\|_{L^2} + \|u_n - u\|_{L^2} \\ &\le 2\|u_n - u\|_{L^2} + \|P_t u_n - u_n\|_{L^2}.\end{aligned}$$

It follows that (P_t) is strongly continuous. □

Let $(L, D(L))$ be the infinitesimal generator of strongly continuous contraction semigroup (P_t) on $L^2(E; m)$. By Theorem 3.3.3, the bilinear form $(\mathscr{E}, \mathscr{F})$ defined by

$$\mathscr{E}(f, g) = (\sqrt{-L} f, \sqrt{-L} g), \ \mathscr{F} = D(\sqrt{-L}), \tag{4.1}$$

is a symmetric form on $L^2(E; m)$ and it may be represented by its approximating form

$$\begin{cases} \mathscr{E}(f, g) = \lim\limits_{t \downarrow 0} \dfrac{1}{t} \langle f - P_t f, g\rangle, \\ \mathscr{F} = \left\{ f \in L^2(E; m) : \sup\limits_t \dfrac{1}{t} \langle f - P_t f, f\rangle < \infty \right\}. \end{cases} \tag{4.2}$$

Recall that we usually write

$$\mathscr{E}^{(t)}(f, g) = \frac{1}{t}\langle f - P_t f, g\rangle, \ \mathscr{E}^{[\beta]}(f, g) = \beta\langle f - \beta U^\beta f, g\rangle.$$

Theorem 4.1.4 $(\mathscr{E}, \mathscr{F})$ is a Dirichlet form on $L^2(E; m)$.

Proof. Though this theorem is actually a consequence of Theorem 3.4.3, we would like to give a direct proof here. It suffices to prove that $(\mathscr{E}, \mathscr{F})$ is Markovian. By symmetry, $m(dx)P_t(x, dy)$ is symmetric and then for $f \in \mathscr{E}^*$,

$$\begin{aligned}
\mathscr{E}^{(t)}(f,f) &= \frac{1}{t}\langle f, f - P_t f\rangle \\
&= \frac{1}{t}\int_E f(x)\left(f(x) - \left(P_t f\right)(x)\right) m(\mathrm{d}x) \\
&= \frac{1}{t}\int_E f(x)\left(f(x) - \int_M f(y)P_t(x,dy)\right) m(dx) \\
&= \frac{1}{t}\int_E u(x)\left(\int_E (u(x) - u(y))P_t(x,dy)\right) m(\mathrm{d}x) + \frac{1}{t}\int_E f^2\left(1 - P_t 1\right)\mathrm{d}m \\
&= \frac{1}{t}\int_{E\times E} f(x)\left(f(x) - f(y)\right) P_t(x,\mathrm{d}y)m(\mathrm{d}x) + \frac{1}{t}\int_E f^2\left(1 - P_t 1\right)\mathrm{d}m \\
&= \frac{1}{2t}\int_{E\times E} \left(f(x) - f(y)\right)^2 P_t(x,\mathrm{d}y)m(\mathrm{d}x) + \frac{1}{t}\int_E f^2\left(1 - P_t 1\right)\mathrm{d}m
\end{aligned}$$

If g is a normal contraction of f, it is then obvious that

$$\mathscr{E}^{(t)}(g,g) \le \mathscr{E}^{(t)}(f,f)$$

and hence $f \in \mathscr{F}$ implies that $g \in \mathscr{F}$ and $\mathscr{E}(g,g) \le \mathscr{E}(f,f)$. □

4.2 Irreducibility and uniqueness of symmetrizing measure

When the process X is symmetric with respect to m, m is called a **symmetrizing measure** of X. The existence and uniqueness of symmetrizing measures of X are always interesting to explore. In this section we shall introduce the notion of fine irreducibility and prove that it implies the uniqueness.

The process X is called *finely irreducible* if $\mathsf{P}^x(T_D < \infty) > 0$ for any $x \in E$ and a non-empty finely open subset D, where T_D is the hitting time of D. Intuitively the fine irreducibility means that any point can reach any non-empty finely open set, while the usual irreducibility means that any point can reach any non-empty open set. Certainly the fine irreducibility is stronger than the usual irreducibility. Since the fine irreducibility is hard to be characterized, we shall give a few equivalent statements which may be useful in some circumstances.

1:0430-1 **Lemma 4.2.1** The following statements are equivalent.
(1) X is finely irreducible.
(2) $U^\alpha 1_D$ is positive everywhere on E for any non-empty finely open set D.
(3) $U^\alpha 1_A$ is either identically zero or positive everywhere on E for any Borel set A or, in other words, $\{U^\alpha(x, \cdot) : x \in E\}$ are all mutually absolutely continuous.
(4) All non-trivial excessive measures are mutually absolutely continuous.

Proof. The equivalence of (1) and (2) is easy. We shall prove that they are equivalent to (3). We may assume $\alpha = 0$. Suppose (1) is true. If $U1_A$ is not identically zero, then there exists $\delta > 0$ such that $D := \{U1_A > \delta\}$ is non-empty. Since $U1_A$ is excessive and thus finely continuous, D is finely open and the fine closure of D is contained in $\{U1_A \geq \delta\}$. Then by Lemma 1.3.7, 1:0430-2

$$U1_A(x) \geq P_D U1_A(x) = \mathsf{E}^x\left(U1_A(X_{T_D})\right) \geq \delta \mathsf{P}^x(T_D < \infty) > 0.$$

Conversely suppose (3) is true. Then for any finely open set D, by the right continuity of X, $U1_D(x) > 0$ for any $x \in D$. Therefore $U1_D$ is positive everywhere on E.

Let ξ be an excessive measure. Since $\alpha\xi U^\alpha \leq \xi$, $\xi(A) = 0$ implies that $\xi U^\alpha(A) = 0$. However ξ is non-trivial. Thus it follows from (3) that $U^\alpha 1_A \equiv$

0, i.e., A is potential zero. Conversely if A is potential zero, then $\xi(A) = 0$ for any excessive measure ξ. Therefore (3) implies (4).

Assume (4) holds. Since $U(x, \cdot)$ is excessive for all x and hence they are equivalent. This implies (3). □

t:0430-1 **Theorem 4.2.2** Assume that X is finely irreducible. Then the symmetrizing measure of X is unique up to a constant. More precisely if both μ and ν are non-trivial symmetrizing measures of X, then $\nu = c\mu$ with a positive constant c.

Proof. First of all there exists a measurable set H such that both $\mu(H)$ and $\nu(H)$ are positive and finite, because μ and ν are equivalent by Lemma l:0430-1 4.2.1. This is actually true when both measures are σ-finite and one is absolutely continuous with respect to another. Indeed, assume that $\nu \ll \mu$. Since ν is non-trivial and σ-finite, we may find a measurable set B such that $0 < \nu(B) < \infty$. Then $\mu(B) > 0$. Since μ is σ-finite, there exist $A_n \uparrow E$ such that $0 < \mu(A_n) < \infty$. Then $\nu(A_n \cap B) \uparrow \nu(B)$ and $\mu(A_n \cap B) \uparrow \mu(B)$. Hence there exists some n such that $\nu(A_n \cap B) > 0$. Take $H = A_n \cap B$, which makes both $\mu(H)$ and $\nu(H)$ positive and finite.

Set $c = \nu(H)/\mu(H)$. We may assume that $c = 1$ without loss of generality. Let $m = \mu + \nu$. Then there is $f_1, f_2 \geq 0$ such $\mu = f_1 \cdot m$ and $\nu = f_2 \cdot m$. Let $A = \{f_1 > f_2\}$, $B = \{f_1 = f_2\}$ and $C = \{f_1 < f_2\}$.

We shall show that $\nu = \mu$. Otherwise $\mu(A) > 0$ or $\nu(C) > 0$. We assume that $\mu(A) > 0$ without loss of generality. Since μ is σ-finite, there is $A_n \in \mathscr{B}(E)$ such that $A_n \subseteq A$, $\mu(A_n) < \infty$ and $A_n \uparrow A$. Let $D = B \cup C$. For any integer n and $\alpha > 0$,

$$(U^\alpha 1_{A_n}, 1_D)_\mu \leq (U^\alpha 1_{A_n}, 1_D)_\nu = (U^\alpha 1_D, 1_{A_n})_\nu \leq (U^\alpha 1_D, 1_{A_n})_\mu.$$

Since $(U^\alpha 1_{A_n}, 1_D)_\mu = (U^\alpha 1_D, 1_{A_n})_\mu$, it follows that $(U^\alpha 1_D, 1_{A_n})_\nu = (U^\alpha 1_D, 1_{A_n})_\mu$. Thus we have

$$(U^\alpha 1_D, (1 - \frac{f_2}{f_1})1_{A_n})_\mu = (U^\alpha 1_D, 1_{A_n})_\mu - (U^\alpha 1_D, 1_{A_n})_\nu = 0.$$

Since $1 - \frac{f_2}{f_1} > 0$ on A, let n go to infinity and by the monotone convergence theorem we get that $(U^\alpha 1_D, 1_A)_\mu = 0$. The fine irreducibility of X implies that $U^\alpha 1_D = 0$ identically or D is of potential zero. Therefore

$$\mu(D) = \nu(D) = 0.$$

Consequently,

$$0 = \mu(H) - \nu(H) = \int_{H\cap A} (1 - \frac{f_2}{f_1}) d\mu$$

which leads to that $\mu(H \cap A) = 0$ and also $\mu(H) = 0$. The contradiction implies that $\nu = \mu$. $\square$

The following example shows that the usual irreducibility is not enough to guarantee the uniqueness of symmetrizing measure, while the fine irreducibility might be too strong.

Example 4.2.3 Let

$$J = \frac{1}{4}(\delta_1 + \delta_{-1} + \delta_{\sqrt{2}} + \delta_{-\sqrt{2}})$$

defined on $\mathbb{R}$ and $\pi = \{\pi_t\}_{t>0}$ the corresponding convolution semigroup; i.e., $\widehat{\pi}_t(x) = e^{-t\phi(x)}$ with

$$\phi(x) = \int (1 - \cos xy) J(dy) = \frac{1}{2}(1 - \cos x) + \frac{1}{2}(1 - \cos \sqrt{2}x).$$

Let X be the corresponding Lévy process. Then X is symmetric with respect to the Lebesgue measure. Let $N = \{n + m\sqrt{2} \ : \ n, m \text{ are integers}\}$ and $\mu = \sum_{x\in N} \delta_x$. Then μ is σ-finite and also a symmetrizing measure. It is easy to see that any point x can reach any point in $x+N$ and can not reach outside of $x+N$. Since $x+N$ is dense in $\mathbb{R}$, any point can reach any non-empty open set, namely X is irreducible. However any compound Poisson process will stay at the starting point for a positive period of time, i.e., any singleton is finely open. Hence X is not finely irreducible.

Another interesting example is also a compound Poisson process X, where the Lévy measure J is a probability measure on $\mathbb{R}$ with a continuous even density. In this case, we can show that X has a unique symmetrizing measure, the Lebesgue measure, but X is still irreducible, while not finely irreducible. Actually any single point can not reach any other point.

It is known that the fine topology is determined by the process and hard to identify usually. Hence it is hard to verify sometimes the fine irreducibility defined in the theorem.

Definition 4.2.4 We say X is **LSC** or **strong Feller**, if $U^\alpha 1_B$ is lower-semi-continuous or continuous, respectively, for any Borel subset B of E.

Lemma 4.2.5 If X is LSC or strong Feller, the fine irreducibility is equivalent to the usual one.

Proof. It suffices to prove that $\mathsf{P}^x(T_D < \infty) > 0$ for any $x \in E$ and non-empty open subset $D \subset E$. In fact, take $A \in \mathscr{B}(E)$ with $U^\alpha 1_A \neq 0$ identically. There is $b > 0$ such that $G = \{U^\alpha 1_A > b\} \neq \emptyset$ and is open due to the property LSC. Again by Lemma 1.3.7, for any $x \in E$,

$$U1_A(x) \geq P_G U1_A(x) = \mathsf{P}^x\left(U1_A(X_{T_G}), T_G < \infty\right).$$

But $X_{T_G} \in \bar{G}$ on $\{T_G < \infty\}$ by Theorem 1.3.5 and then $X_{T_G} \geq b$ on $\{T_G < \infty\}$. Hence by the irreducibility, we have

$$U1_A(x) \geq b\mathsf{P}^x\left(T_G < \infty\right) > 0.$$

□

Question If X is a Lévy process, what conditions imposed on its Lévy exponent guarantee that X is irreducible or finely irreducible?

4.3 Restriction